§ 3.2 Stationary distributions

Def A probability vector π is called a stationary distribution for a Markov chain with transition matrix P if $\pi = \pi P$.

Remark Suppose π is a stationary distribution and $X_0 \sim \pi$ (ie. π is the initial distribution of the chain). Then $X_1 \sim \pi P = \pi$, $X_2 \sim \pi P^2 = \pi P = \pi$ and by induction $X_n \sim \pi P^n = \pi$ for all $n \geq 0$.

Lemma If λ is a limiting distribution then it's a stationary distribution.

Proof Let α be any initial distribution of the Markov chain and denote the distribution of X_n by α_n for all $n \geq 1$. Note that

$$\lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \alpha P^n = \lambda$$

Moreover, $\alpha_n = \alpha_{n-1} P$ for any $n \geq 1$, and taking the limit of both sides we get

$$\lambda = \lim_{n \to \infty} \alpha_n = \left(\lim_{n \to \infty} \alpha_{n-1}\right) P = \lambda P.$$

<u>Warning</u> If π is a stationary distribution it's not necessarily a limiting distribution (there a Markov chains that have a stationary distribution but don't have a limiting distribution)

<u>Example</u> Consider a random walk on a graph $G=(V,E)$. So its state space is the set of vertices V and for each $v,w \in V$,

$$P_{vw} = \begin{cases} \frac{1}{\deg(v)} & \text{if } v \sim w \\ 0 & \text{if } v \sim w \end{cases}$$

where $v \sim w$ means v and w are connected by an edge and $\deg(v)$ is the number of edges eminating from V. Then the probability vector π where $\pi_v = \frac{\deg(v)}{2|E|}$ for each $v \in V$ is a stationary distribution.

$$(\pi P)_v = \sum_{w \in V} \pi_w P_{wv} = \sum_{w \sim v} \pi_w P_{wv}$$

$$= \sum_{w \sim v} \left(\frac{\deg(w)}{2|E|} \right) \frac{1}{\deg(w)}$$

$$= \frac{1}{2|E|} \sum_{w \sim v} 1 = \frac{1}{2|E|} \deg(v) = \pi_v$$

In linear algebra, we learn to solve $Ax=b$ by making an augmented matrix $[A \,\vdots\, b]$ and putting it in reduced row echelon form to find the solution set.

Finding stationary distributions is an algebraic problem: we have to solve the linear system of equations

$$\begin{cases} \pi P = \pi \\ \pi_1 + \cdots + \pi_m = 1 \end{cases} \iff \begin{cases} \pi(P-I) = 0 \\ \pi_1 + \cdots + \pi_m = 1 \end{cases} \iff \begin{cases} (P-I)^T \begin{bmatrix} \pi_1 \\ \vdots \\ \pi_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \\ \pi_1 + \cdots + \pi_m = 1 \end{cases}$$

We do this by find the RREF of

$$\left[\begin{array}{ccc|c} & (P-I)^T & & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ 1 & \cdots & 1 & 1 \end{array}\right]$$

Problem 1. Consider the Markov chain with transition matrix

$$P = \begin{bmatrix} 1/4 & 3/4 \\ 2/3 & 1/3 \end{bmatrix}.$$

a. It is not always the case that a Markov chain has a limiting distribution, but if it does, we've learned that the limiting distribution must be a stationary distribution. That is, it must satisfy the algebraic equation $\pi P = \pi$ where π is a row vector with two components that sum to 1. This yields a system with 2 unknowns (the two components of π) and 3 equations (why 3?). Write the system of 3 equations out and then express it as an augmented matrix.

b. Use the `rref.Rmd` file on the class webpage to find the *reduced row echelon form* of the augmented matrix you found. Convert what you get back into equations involving π_1 and π_2.

c. Compare your solution to what you get using the formula we learned last time for the limiting distribution of a 2-state Markov chain.

$$(\pi_1, \pi_2)\begin{pmatrix} 1/4 & 3/4 \\ 2/3 & 1/3 \end{pmatrix} = (\pi_1, \pi_2) \qquad (\text{and } \pi_1 + \pi_2 = 1)$$

$$\Rightarrow \left(\tfrac{1}{4}\pi_1 + \tfrac{2}{3}\pi_2, \tfrac{3}{4}\pi_1 + \tfrac{1}{3}\pi_2\right) = (\pi_1, \pi_2)$$

$$\Rightarrow \begin{cases} \tfrac{1}{4}\pi_1 + \tfrac{2}{3}\pi_2 = \pi_1 \\ \tfrac{3}{4}\pi_1 + \tfrac{1}{3}\pi_2 = \pi_2 \end{cases} \Rightarrow \begin{cases} -\tfrac{3}{4}\pi_1 + \tfrac{2}{3}\pi_2 = 0 \\ \tfrac{3}{4}\pi_1 - \tfrac{2}{3}\pi_2 = 0 \\ \pi_1 + \pi_2 = 1 \end{cases}$$

$$\Rightarrow \left[\begin{array}{cc:c} -3/4 & 2/3 & 0 \\ 3/4 & -2/3 & 0 \\ 1 & 1 & 1 \end{array}\right] \underset{\text{with R}}{\overset{\text{RREF}}{\rightsquigarrow}} \left[\begin{array}{cc:c} 1 & 0 & 8/17 \\ 0 & 1 & 9/17 \\ 0 & 0 & 0 \end{array}\right]$$

notice this is the same as $\left[\begin{array}{c:c} (P-I)^T & \begin{matrix} 0 \\ 0 \end{matrix} \\ 1 \quad 1 & 1 \end{array}\right]$

$$\Rightarrow \begin{cases} \pi_1 = 8/17 \\ \pi_2 = 9/17 \\ 0 = 0 \end{cases}$$

So $(8/17, 9/17)$ is (the unique) stationary distribution

Problem 2. Consider the Markov chain with transition matrix

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/3 & 2/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find all stationary distributions of P by solving $\pi P = \pi$. Use R to find the reduced row echelon form of an appropriate augmented matrix. Remember that sometimes a linear system of equations has infinitely many solutions and you need to introduce the notion of free variables to parametrize the solution set.

π_3 is "free variable"

$$\left[\begin{array}{ccc|c} & (P-I)^T & & \begin{matrix}0\\0\\0\end{matrix} \\ 1 & 1 & 1 & 1 \end{array}\right] \xrightarrow[\text{in R}]{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 2/5 & 2/5 \\ 0 & 1 & 3/5 & 3/5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} \pi_1 + 2/5\, \pi_3 = 2/5 \\ \pi_2 + 3/5\, \pi_3 = 3/5 \\ \pi_3 = t \end{cases}$$

$\Rightarrow$ Infinite set of stationary distributions of the form

$$\left\{ \left(\frac{2}{5} - \frac{2}{5} t, \frac{3}{5} - \frac{3}{5} t, t \right) : 0 \le t \le 1 \right\}$$

Problem 3. Repeat the previous problem using

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 4/5 & 1/5 \\ 0 & 0 & 1/4 & 3/4 \end{bmatrix}.$$

π_4 is free variable

$$\left[\begin{array}{cccc|c} & (P-I)^T & & & \begin{matrix}0\\0\\0\\0\end{matrix} \\ 1 & 1 & 1 & 1 & 1 \end{array}\right] \xrightarrow[\text{in R}]{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 9/10 & 2/5 \\ 0 & 1 & 0 & 27/20 & 3/5 \\ 0 & 0 & 1 & -5/4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} \pi_1 + \frac{9}{10} \pi_4 = 2/5 \\ \pi_2 + \frac{27}{20} \pi_4 = 3/5 \\ \pi_3 = 5/4\, \pi_4 \\ \pi_4 = t \end{cases} \Rightarrow$$

Infinite set of stationary distribution of the form

to ensure prob. vector

$$\left\{ \left(\frac{2}{5} - \frac{9}{10} t, \frac{3}{5} - \frac{27}{20} t, \frac{5}{4} t, t \right) : 0 \le t \le \frac{4}{9} \right\}$$

Problem 4. Use the same algebraic technique we have used in the rest of this worksheet to find all stationary distributions of the random walk on a 6-cycle. Compare with the general formula we learned for the stationary distribution of a random walk on a graph. Explain the advantage of the algebraic technique using the terms *existence* and *uniqueness* in your explanation.

We proved that $\pi_v = \dfrac{\deg(v)}{2|E|}$ always gives a stationary distribution for a random walk on a graph (so we showed existence of a stationary distribution) but this does not address whether there are other stationary distributions (it does not address uniqueness). The algebraic technique finds all stationary distributions, so it addresses existence and uniqueness.