§ 3.3 Classification of states

Def State j is said to be accessible from state i if there exists $n \geq 0$ such that $P_{ij}^{n} > 0$. States i and j are said to communicate if they are accessible from each other. We denote this by $i \sim j$.

Remark Communication is what is known in mathematic as an equivalence relation, meaning it satisfies:

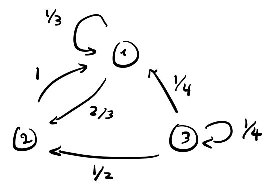
1. reflexivity: $i \sim i$
2. symmetry: $i \sim j$ implies $j \sim i$
3. transitivity: if $i \sim j$ and $j \sim k$, then $i \sim k$.

As a result, the sample space can be partitioned into disjoint sets, called communication classes, of states that communicate among themselves.

If all states in the state space communicate with each other (i.e. there is only one communication class) then the Markov chain is said to be irreducible.

Def A state i is said to be recurrent if, given that the chain starts at i, the conditional probability the chain eventually returns to i is 1. Otherwise (i.e. if the prob. of eventually returning is less than 1), the state is called transient.

<u>Example</u> Consider the following Markov chain. Find the communication classes and classify the states as recurrent or transient.



The classes are $\{1,2\}, \{3\}$.

Let $f_j = P(X_n = j \text{ for some } n \geq 1 \mid X_0 = j)$ for each $j = 1, 2, 3$.

State 3 is transient since

$$f_3 = 1 - P(X_n \neq 3 \text{ for all } n \geq 1 \mid X_0 = 3)$$

$$= 1 - \left(\tfrac{1}{4} + \tfrac{1}{2}\right) = \tfrac{1}{4} < 1$$

State 1 is recurrent since we are guaranteed to return to 1 in 1 or 2 steps. That is,

$$f_1 \geq P(X_1 = 1 \mid X_0 = 1) + P(X_2 = 1, X_1 = 2 \mid X_0 = 1)$$

$$= \tfrac{1}{3} + \tfrac{2}{3} = 1, \quad \text{so } f_1 = 1,$$

since $\{X_1 = 1\} \cup \{X_2 = 1, X_1 = 2\} \subseteq \{X_n = 1 \text{ for some } n \geq 1\}$.

State 2 is recurrent since

$$f_2 = 1 - P(X_n \neq 2 \text{ for all } n \geq 1 \mid X_0 = 2)$$

$$= 1 - P(X_n = 1 \text{ for all } n \geq 1 \mid X_0 = 2)$$

$$= 1 - \lim_{n \to \infty} \left(\tfrac{1}{3}\right)^n$$

$$= 1$$

Canonical Decomposition

Suppose our state space S is decomposed as

$$S = T \cup R_1 \cup R_2 \cup \cdots \cup R_m$$

where T is the set of all transient states and $R_1, R_2, \ldots, R_m$ are communication classes of recurrent states (we will soon learn that states in a given communication class are either all transient or all recurrent). Given this decomposition, called the <u>canonical</u> <u>decomposition</u> of S, we can reorder the entries of the transition matrix P:

$$\begin{array}{c} \\ T \\ R_1 \\ R_2 \\ \vdots \\ R_m \end{array} \begin{array}{c} \begin{array}{ccccc} T & R_1 & R_2 & \cdots & R_m \end{array} \\ \left(\begin{array}{ccccc} & & & & \\ & P_1 & & & \\ & & P_2 & & \\ & & & \ddots & \\ & & & & P_m \end{array} \right) \end{array}$$

So that the blocks $P_1, P_2, \ldots, P_m$ are transition probabilities among states of the respective communication classes.

Problem 1. For each of the transition matrices below, draw the transition state diagram for the chain and then find the communication classes. That is, partition the state space into disjoint sets of states that communicate with each other.

$$P_1 = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}, P_2 = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 0 & 0 & 1/2 \end{bmatrix},$$

$$P_3 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1/3 & 0 & 0 & 2/3 & 0 & 0 \\ 0 & 1/4 & 0 & 0 & 3/4 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 1/7 & 0 & 0 & 6/7 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 2/3 \end{bmatrix} \end{array}.$$

Problem 2. We will soon learn that for any communication class, states in that class are either all recurrent or all transient. For each communication class you found in the Markov chains above, decide whether its states are transient or recurrent. No need to give detailed justification.

P_1 $\{1,2\}$, $\{4\}$ recurrent classes, $\{3\}$ transient class

P_2 $\{1,2\}$, $\{3,4\}$ recurrent, $\{5\}$ transient

P_3 $\{1,4\}$, $\{2,5\}$ recurrent, $\{3\}$, $\{6\}$ transient

Problem 3. Discuss the following question with group mates. For transition matrix P_3 you should find that the communication classes are $\{1,4\}$, $\{2,5\}$, $\{3\}$, and $\{6\}$. Suppose we rearrange the rows and columns so that states in the same communication class are adjacent as below. Fill in the entries in the matrix below. What do you notice about the block structure? Discuss how you think this rearranging of states (what we the *canonical decomposition* of the state space) and reconfiguring of the matrix P_3 (what we call the *canonical form* of P_3) helps you understand $\lim_{n\to\infty} P_3^n$.

$$P_3 = \begin{array}{c} \\ 3 \\ 6 \\ 1 \\ 4 \\ 2 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 3 & 6 & 1 & 4 & 2 & 5 \end{array} \\ \begin{bmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} \end{array}.$$

$$\begin{array}{c} \\ 3 \\ 6 \\ 1 \\ 4 \\ 2 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 3 & 6 & 1 & 4 & 2 & 5 \end{array} \\ \begin{bmatrix} 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 2/3 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 1/7 & 6/7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/4 & 3/4 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix} \end{array}$$

Problem 4. Discuss the following question with group mates. Suppose you have a communication class C that consists of all recurrent states. Let $i \in C$ and $j \notin C$. Make a conjecture for what you can conclude about P_{ij} and give an explanation for your conjecture. This tells us more about the structure of the canonical form of P in general (not just the examples above). How so?

See Lemma 3.5