

§ 3.3 More on recurrence and transience

Consider repeating trials of a success/failure experiment until we get our first success (eg. repeatedly toss a coin until you get heads). If we let \bar{X} count the number of trials performed, up to and including the successful trial, then

$$P(\bar{X} = k) = (1-p)^{k-1} p^k \text{ for } k=1, 2, 3, \dots$$

where p denotes the probability of success. That is,

$\bar{X} \sim \text{Geom}(p)$, is geometrically distributed and $E[\bar{X}] = \frac{1}{p}$.

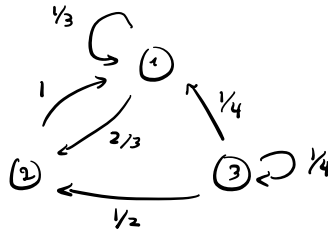
Def If we run a simulation of a Markov chain, the resulting set of states visited is called a sample path.

An excursion from state i is a piece of the sample path that starts and ends at i . If the chain starts at i , upon each return, we say the Markov chain regenerates,

because it's as if it's starting again from time 0.

Note that excursions yield independent segments of the sample path.

Example Consider the Markov chain



We learned last time that state 3 is transient since $f_3 = P(\bar{X}_n = 3 \text{ for some } n \geq 1 \mid \bar{X}_0 = 3) < 1$. We also saw $f_1 = f_2 = 1$, making 1 and 2 recurrent states.

Let N_j count the number of visits to j over the entire history of the Markov chain. Then, for example, N_3 is helping count excursions the Markov chain makes to state 3 before it leaves and never returns. Notice

$$P(N_3 = 1 \mid \bar{X}_0 = 3) = \frac{3}{4} = 1 - f_3$$

$$P(N_3 = 2 \mid \bar{X}_0 = 3) = \frac{1}{4} \left(\frac{3}{4}\right) = f_3 (1 - f_3)$$

$$P(N_3 = 3 \mid \bar{X}_0 = 3) = \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right) = f_3^2 (1 - f_3)$$

and $N_3 \mid \bar{X}_0 = 3 \sim \text{Geom}(1 - f_3)$.

The situation with states 1 and 2 is different. Here,

$N_1 = N_2 = \infty$ almost surely (ie. we are guaranteed, with probability 1, to return to 1 and 2 infinitely often).

Theorem Let $j \in S$ be a given state in a Markov chain with transition matrix P . Then

① j is transient if and only if $\sum_{n=0}^{\infty} P_{jj}^n < \infty$

② j is recurrent if and only if $\sum_{n=0}^{\infty} P_{jj}^n = \infty$.

Proof in worksheet.

Problem 1. Consider a Markov chain with state j and let f_j denote the conditional probability that the chain eventually returns to state j given that $X_0 = j$.

a. Suppose j is a transient state.

- Let N denote the number of visits the Markov chain makes to state j over its whole history. What is the conditional distribution of N given $X_0 = j$? Before answering, give a general expression for $P(N = k | X_0 = j)$ for any $k \geq 1$.
- Find $E[N | X_0 = j]$.

b. Suppose j is a recurrent state. Find $E[N | X_0 = j]$.

① $N | \bar{X}_0 = j \sim \text{Geom}(1 - f_j), E[N | \bar{X}_0 = j] = \frac{1}{1 - f_j}$

② $E[N | \bar{X}_0 = j] = \infty$

Problem 2. Let

$$I_n = \begin{cases} 1 & \text{if } X_n = j \\ 0 & \text{if } X_n \neq j \end{cases}$$

be an indicator random variable which tells us whether the Markov chain is in a given state j at time n .

a. Find a formula for N that expresses it in terms of I_0, I_1, I_2, \dots

b. Compute $\sum_{n=0}^{\infty} E[I_n | X_0 = j]$ and explain why

1. j is transient if and only if

$$\sum_{n=0}^{\infty} P_{jj}^n < \infty.$$

2. j is recurrent if and only if

$$\sum_{n=0}^{\infty} P_{jj}^n = \infty.$$

① $N = \sum_{n=0}^{\infty} I_n$

② $\sum_{n=0}^{\infty} P_{jj}^n = \sum_{n=0}^{\infty} E[I_n | \bar{X}_0 = j] = E\left[\sum_{n=0}^{\infty} I_n | \bar{X}_0 = j\right] = E[N | \bar{X}_0 = j]$

$$= \begin{cases} \infty & \text{iff } j \text{ is recurrent} \\ \frac{1}{1 - f_j} & \text{iff } j \text{ is transient} \end{cases}$$

Theorem (recurrence and transience are class properties)

The states of a communication class are either all recurrent or all transient.

Proof Let i and j be states in the same communication class. Suppose i is recurrent. We must show j is recurrent.

That is, we must show $\sum_{n=0}^{\infty} P_{jj}^n = \infty$. Since i and

j communicate, there exist $r, s \geq 0$ such that

$P_{ij}^r > 0$ and $P_{ji}^s > 0$. Let $n \geq 0$. Then

$$\begin{aligned} P_{jj}^{r+s+n} &= P(\bar{X}_{r+s+n}=j \mid \bar{X}_0=j) \\ &\geq P(\bar{X}_s=i, \bar{X}_{s+n}=i, \bar{X}_{r+s+n}=j \mid \bar{X}_0=j) \\ &= P_{ji}^s P_{ii}^n P_{ij}^r. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{n=0}^{\infty} P_{jj}^{r+s+n} &\geq \sum_{n=0}^{\infty} P_{ji}^s P_{ii}^n P_{ij}^r \\ &= P_{ji}^s \left(\sum_{n=0}^{\infty} P_{ii}^n \right) P_{ij}^r = \infty \end{aligned}$$

which implies $\sum_{n=0}^{\infty} P_{jj}^n = \infty$ since $\sum_{n=0}^{\infty} P_{jj}^n = \sum_{n=0}^{r+s-1} P_{jj}^n + \sum_{n=0}^{\infty} P_{jj}^{r+s+n}$.

Now suppose i is transient. Then j must be transient

too. Otherwise, if j were recurrent, then we'd be able

to prove i was recurrent too by repeating the previous

argument