§ 3.3 More on recurrence and transience

Consider repeating trials of a success/failure experiment until we get our first success (e.g. repeatedly toss a coin until you get heads). If we let X count the number of trials peformed, up to and including the successful trial, then

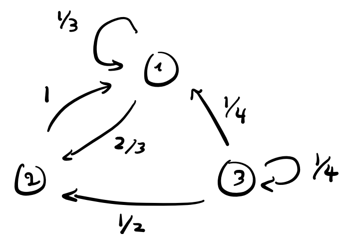
$$P(X = k) = (1-p)^{k-1} p^{k} \quad \text{for } k = 1, 2, 3, \ldots$$

where p denotes the probability of success. That is, $X \sim \text{Geom}(p)$, is geometrically distributed and $E[X] = \frac{1}{p}$.

<u>Def</u> If we run a simulation of a Markov chain, the resulting set of states visited is called a <u>sample path</u>. An <u>excursion</u> from state i is a piece of the sample path that starts and ends at i. If the chain starts at i, upon each return, we say the Markov chain <u>regenerates</u>, because it's as if it's starting again from time 0.

Note that excursions yield independent segments of the sample path.

Example Consider the Markov chain



We learned last time that state 3 is transient since $f_3 = P(X_n = 3 \text{ for some } n \geq 1 \mid X_0 = 3) < 1$. We also saw $f_1 = f_2 = 1$, making 1 and 2 recurrent states.

Let N_j count the number of visits to j over the entire history of the Markov chain. Then, for example, N_3 is helping count excursions the Markov chain makes to state 3 before it leaves and never returns. Notice

$$P(N_3 = 1 \mid X_0 = 3) = \frac{3}{4} = 1 - f_3$$
$$P(N_3 = 2 \mid X_0 = 3) = \frac{1}{4}\left(\frac{3}{4}\right) = f_3(1 - f_3)$$
$$P(N_3 = 3 \mid X_0 = 3) = \left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right) = f_3^2(1 - f_3)$$

and $N_3 \mid X_0 = 3 \sim \text{Geom}(1 - f_3)$.

The situation with states 1 and 2 is different. Here, $N_1 = N_2 = \infty$ almost surely (i.e. we are guaranteed, with probability 1, to return to 1 and 2 infinitely often).

Theorem Let $j \in S$ be a given state in a Markov chain with transition matrix P. Then

① j is transient if and only if $\sum_{n=0}^{\infty} P_{jj}^{n} < \infty$

② j is recurrent if and only if $\sum_{n=0}^{\infty} P_{jj}^{n} = \infty$.

Proof in worksheet.

Problem 1. Consider a Markov chain with state j and let f_j denote the conditional probability that the chain eventually returns to state j given that $X_0 = j$.

a. Suppose j is a transient state.

1. Let N denote the number of visits the Markov chain makes to state j over its whole history. What is the conditional distribution of N given $X_0 = j$? Before answering, give a general expression for $P(N = k \mid X_0 = j)$ for any $k \geq 1$.
2. Find $E[N \mid X_0 = j]$.

b. Suppose j is a recurrent state. Find $E[N \mid X_0 = j]$.

ⓐ $N \mid X_0 = j \sim \text{Geom}(1 - f_j)$, $E[N \mid X_0 = j] = \frac{1}{1 - f_j}$

ⓑ $E[N \mid X_0 = j] = \infty$

Problem 2. Let

$$I_n = \begin{cases} 1 & \text{if } X_n = j \\ 0 & \text{if } X_n \neq j \end{cases}$$

be an indicator random variable which tells us whether the Markov chain is in a given state j at time n.

a. Find a formula for N that expresses it in terms of $I_0, I_1, I_2, \ldots$.

b. Compute $\sum_{n=0}^{\infty} E[I_n \mid X_0 = j]$ and explain why

1. j is transient if and only if
$$\sum_{n=0}^{\infty} P_{jj}^{n} < \infty.$$
2. j is recurrent if and only if
$$\sum_{n=0}^{\infty} P_{jj}^{n} = \infty.$$

ⓐ $N = \sum_{n=0}^{\infty} I_n$

ⓑ $\sum_{n=0}^{\infty} P_{jj}^{n} = \sum_{n=0}^{\infty} E[I_n \mid X_0 = j] = E\left[\sum_{n=0}^{\infty} I_n \mid X_0 = j\right] = E[N \mid X_0 = j]$

$$= \begin{cases} \infty & \text{iff } j \text{ is recurrent} \\ \frac{1}{1 - f_j} & \text{iff } j \text{ is transient} \end{cases}$$

Theorem (recurrence and transience are class properties)
The states of a communication class are either all recurrent or all transient.

Proof Let i and j be states in the same communication class. Suppose i is recurrent. We must show j is recurrent. That is, we must show $\sum_{n=0}^{\infty} P_{jj}^{n} = \infty$. Since i and j communicate, there exist $r, s \geq 0$ such that $P_{ij}^{r} > 0$ and $P_{ji}^{s} > 0$. Let $n \geq 0$. Then

$$
\begin{aligned}
P_{jj}^{r+s+n} &= P(X_{r+s+n}=j \mid X_0 = j) \\
&\geq P(X_s = i, X_{s+n} = i, X_{r+s+n} = j \mid X_0 = j) \\
&= P_{ji}^{s} P_{ii}^{n} P_{ij}^{r}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sum_{n=0}^{\infty} P_{jj}^{r+s+n} &\geq \sum_{n=0}^{\infty} P_{ji}^{s} P_{ii}^{n} P_{ij}^{r} \\
&= P_{ji}^{s} \left(\sum_{n=0}^{\infty} P_{ii}^{n} \right) P_{ij}^{r} = \infty
\end{aligned}
$$

which implies $\sum_{n=0}^{\infty} P_{jj}^{n} = \infty$ since $\sum_{n=0}^{\infty} P_{jj}^{n} = \sum_{n=0}^{r+s-1} P_{jj}^{n} + \sum_{n=0}^{\infty} P_{jj}^{r+s+n}$.

Now suppose i is transient. Then j must be transient too. Otherwise, if j were recurrent, then we'd be able to prove i was recurrent too by repeating the previous argument

Theorem If a Markov chain is irreducible and has a finite state space, then all states are recurrent.

Proof Let $S = \{1, 2, \ldots, m\}$ denote the Markov chain's state space. Since the chain is irreducible, it has one communication class and so either all states are recurrent or all states are transient. Suppose they're all transient. Then if the chain starts in state 1, it must eventually not return and will live among states $2, 3, \ldots, m$. But then the chain will eventually never return to state 2 and live only among $3, \ldots, m$. Eventually, since the state space is finite, there will be no states left that can be visited.