Math 339SP — Communication classes

Problem 1. For each of the transition matrices below, draw the transition state diagram for the chain and then find the communication classes. That is, partition the state space into disjoint sets of states that communicate with each other.

$$P_1 = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}, P_2 = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 0 & 0 & 1/2 \end{bmatrix},$$

$$P_3 = \begin{array}{c} \scriptstyle 1 \\ \scriptstyle 2 \\ \scriptstyle 3 \\ \scriptstyle 4 \\ \scriptstyle 5 \\ \scriptstyle 6 \end{array} \overset{\begin{array}{cccccc} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 & \scriptstyle 4 & \scriptstyle 5 & \scriptstyle 6 \end{array}}{\begin{bmatrix} 1/3 & 0 & 0 & 2/3 & 0 & 0 \\ 0 & 1/4 & 0 & 0 & 3/4 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 1/7 & 0 & 0 & 6/7 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 2/3 \end{bmatrix}}.$$

Problem 2. We will soon learn that for any communication class, states in that class are either all recurrent or all transient. For each communication class you found in the Markov chains above, decide whether its states are transient or recurrent. No need to give detailed justification.

Problem 3. Discuss the following question with group mates. For transition matrix P_3 you should find that the communication classes are $\{1,4\}$, $\{2,5\}$, $\{3\}$, and $\{6\}$. Suppose we rearrange the rows and columns so that states in the same communication class are adjacent as below. Fill in the entries in the matrix below. What do you notice about the block structure? Discuss how you think this rearranging of states (what we the *canonical decomposition* of the state space) and reconfiguring of the matrix P_3 (what we call the *canonical form* of P_3) helps you understand $\lim_{n\to\infty} P_3^n$.

$$P_3 = \begin{array}{c} \scriptstyle 3 \\ \scriptstyle 6 \\ \scriptstyle 1 \\ \scriptstyle 4 \\ \scriptstyle 2 \\ \scriptstyle 5 \end{array} \overset{\begin{array}{cccccc} \scriptstyle 3 & \scriptstyle 6 & \scriptstyle 1 & \scriptstyle 4 & \scriptstyle 2 & \scriptstyle 5 \end{array}}{\begin{bmatrix} & & & & & \\ \\ \\ \\ \\ \\ \end{bmatrix}}.$$

Problem 4. Discuss the following question with group mates. Suppose you have a communication class C that consists of all recurrent states. Let $i \in C$ and $j \notin C$. Make a conjecture for what you can conclude about P_{ij} and give an explanation for your conjecture. This tells us more about the structure of the canonical form of P in general (not just the examples above). How so?