Math 339SP, Spring 2022 — Excursions

Class on February 17

An *excursion* of a Markov chain is a chunk of its path that starts and ends at the same state. For example, if a Markov chain starts at state j at time 0, visits other states, and then eventually returns to state j at time n, this is called an excursion of length n from state j. Upon each return to state j we say that a *regeneration* has occurred, which informally means that it is probabilistically equivalent to think of the chain as starting from time 0 again. More formally, if T denotes the random time when the Markov chain first returns to state j, the sequence

$$X_T, X_{T+1}, \ldots$$

is a Markov chain and is independent of $X_0, X_1 \ldots, X_{T-1}$.

Problem 1. Consider a Markov chain with state j and let f_j denote the conditional probability that the chain eventually returns to state j given that $X_0 = j$.

1. Suppose j is a transient state.
   (a) What do we know about f_j?
   (b) Let N denote the number of visits the Markov chain makes to state j over its whole history. What is the conditional distribution of N given $X_0 = j$? Before answering this, try finding $P(N = k \mid X_0 = j)$ for $k = 1, 2, 3$.
   (c) Find $E[N \mid X_0 = j]$.
2. Suppose j is a recurrent state. Find $E[N \mid X_0 = j]$.

Problem 2. Let

$$I_n = \begin{cases} 1 & \text{if } X_n = j \\ 0 & \text{if } X_n \neq j \end{cases}$$

be an indicator random variable which tells us whether the Markov chain is in a given state j at time n.

1. Explain why $N = \displaystyle\sum_{n=0}^{\infty} I_n$.
2. Take the conditional expectation, given $X_0 = j$, of both sides above and explain why
   (a) j is transient if and only if
   $$\sum_{n=0}^{\infty} P_{jj}^n < \infty.$$
   (b) j is recurrent if and only if
   $$\sum_{n=0}^{\infty} P_{jj}^n = \infty.$$