Math 342, Fall 2024 — Homework 10

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Due December 6 at 5:00 pm

Instructions. This problem set contains problems from Week 12 of class. The problem numbers refer to our textbook, *Probability with Applications and R*, by Amy Wagaman and Robert Dobrow, 2nd edition.

Problem 1. Do the following textbook problems and submit on Gradescope: 9.2, 9.4, 9.10, 9.14.

Problem 2. Let S be the region in $\mathbb{R}^2$ whose boundary is the triangle with vertices at $(0,0), (1,0), (1,2)$. Suppose that X and Y are random variables with joint density given by

$$f(x,y) = \begin{cases} \frac{5}{2}x^2 y & (x,y) \in S \\ 0 & \text{otherwise.} \end{cases}$$

a. Find the marginal density of Y.

b. Find the conditional density of X given $Y = y$.

c. Find $P(X < 3/4 \mid Y = 1)$.

Problem 3. Let X and Y have joint density

$$f(x,y) = \begin{cases} e^{-x(y+1)} & x > 0, 0 < y < e - 1 \\ 0 & \text{otherwise.} \end{cases}$$

a. Find the marginal density of Y.

b. Find the conditional density of X given $Y = y$.

c. Find $P(4 < X < 5 \mid Y = 1)$.

d. Find $E[X \mid Y = y]$.

e. Find $E[X]$.

Problem 4. You're given two chances to take a certain exam. On the first try, suppose that your score X is uniformly distributed on the interval $(0, 100)$. Further, suppose that on the second try, you're guaranteed to score at least as well as on the first try and your score Y is uniformly distributed on the interval from your first score to 100.

a. Given that your score on the second try was 90, find the probability that you scored less than 50 on the first try.

b. Find $E[Y]$.

Problem 5. Suppose that in a certain population, the number of traffic accidents a person is involved in is a Poisson distributed random variable X. However, suppose that the parameter is unknown. As such, since outside experts believe most of the population has between 2 and 4 accidents per year, the parameter is modeled by a continuous random variable $\Lambda \sim \text{Unif}(2,4)$.

a. Find $E[X \mid \Lambda = \lambda]$.

b. Find $E[X \mid \Lambda]$.

c. Find $E[X]$.

Problem 6. In many modeling scenarios, we're interested in computing a probability of the form $P(Y \leq a)$ or $P(Y = a)$ where a is a given constant and Y is a continuous or discrete random variable whose distribution is not given explicitly. Instead, we might only know the marginal distribution of X and the conditional distribution of Y given $X = x$. We've seen a few examples like this. It's possible to compute probabilities like these by a method called *conditioning* on X, which works like a continuous version of the law of total probability:

$$P(Y \leq a) = \int_{-\infty}^{\infty} P(Y \leq a \mid X = x) f_X(x)\, dx,$$

$$P(Y = a) = \int_{-\infty}^{\infty} P(Y = a \mid X = x) f_X(x)\, dx.$$

Use the method of conditioning to do the following problems.

a. Let $X \sim \text{Unif}(0,1)$. Given $X = x$, the conditional distribution of Y is exponential with parameter x. Find $P(Y > 1)$.

b. Suppose that the probability a certain person gets at least one cold in a year is unknown and modeled as a random variable X whose density is $12x^2(1-x)$ for $0 < x < 1$. Find the probability that over the next 10 years, this person contracts at least one cold in exactly 9 of these years. You should assume that the probability of contracting at least one cold in a year stays the same throughout the 10 years and that each year is independent.

Problem 7. If you liked the problems above or want more practice, our textbook has more great problems. The odd-numbered ones have solutions in the back. Here are some that I recommend (as optional, not to be turned in): 9.1, 9.3, 9.8, 9.15, 9.19, 9.23, 9.25. Feel free to try others, including all the problems in the main sections, which include full explanations.