Math 342, Spring 2024 — Homework 10

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Due April 26 at 5:00 pm

Instructions. This problem set contains problems from Week 13 of class. The problem numbers refer to our textbook, *Probability with Applications and R*, by Amy Wagaman and Robert Dobrow, 2nd edition.

Problem 1. Do the following textbook problems and submit on Gradescope: 9.2.

Problem 2. Let S be the region in $\mathbb{R}^2$ whose boundary is the triangle with vertices at $(0,0),(1,0),(1,2)$. Suppose that X and Y are random variables with joint density given by

$$f(x,y) = \begin{cases} \frac{5}{2}x^2 y & (x,y) \in S \\ 0 & \text{otherwise.} \end{cases}$$

a. Find the marginal density of Y.

b. Find the conditional density of X given $Y = y$.

c. Find $P(X < 3/4 \mid Y = 1)$.

Problem 3. Let X and Y have joint density

$$f(x,y) = \begin{cases} e^{-x(y+1)} & x > 0, 0 < y < e-1 \\ 0 & \text{otherwise.} \end{cases}$$

a. Find the marginal density of Y.

b. Find the conditional density of X given $Y = y$.

c. Find $P(4 < X < 5 \mid Y = 1)$.

d. Find $E[X \mid Y = y]$.

e. Find $E[X]$.

Problem 4. You're given two chances to take a certain exam. On the first try, suppose that your score X is uniformly distributed on the interval $(0,100)$. Further, suppose that on the second try, you're guaranteed to score at least as well as on the first try and your score Y is uniformly distributed on the interval from your first score to 100.

a. Given that your score on the second try was 90, find the probability that you scored less than 50 on the first try.

b. Find $E[Y]$.

Problem 5. Suppose that in a certain population, 60% of people live in region A and 40% live in region B. Those who live in region A have on average 2 traffic accidents per year while those who live in region B have on average 4 traffic accidents per year. Let X be a random variable with probability mass function

$$P(X = x) = \begin{cases} 0.6 & x = 2 \\ 0.4 & x = 4 \end{cases}$$

If we choose a person at random from the population, the number of accidents Y that person has in a year is a random variable which is, conditional on $X = x$, Poisson distributed with parameter x.

a. Find $E[Y]$.

b. Suppose that the population's average number of accidents X doesn't have such a discrete breakdown based on the two regions. Instead, suppose that the average number of accidents is uniformly distributed in the interval $(2, 4)$. Find $E[Y]$ in this case.

Problem 6. In many modeling scenarios, we're interested in computing a probability of the form $P(Y \leq a)$ or $P(Y = a)$ where a is a given constant and Y is a continuous or discrete random variable whose distribution is not given explicitly. Instead, we might only know the marginal distribution of X and the conditional distribution of Y given $X = x$. We've seen a few examples like this. It's possible to compute probabilities like these by a method called *conditioning* on X, which works like a continuous version of the law of total probability:

$$P(Y \leq a) = \int_{-\infty}^{\infty} P(Y \leq a \mid X = x) f_X(x)\, dx,$$
$$P(Y = a) = \int_{-\infty}^{\infty} P(Y = a \mid X = x) f_X(x)\, dx.$$

Use the method of conditioning to do the following problems.

a. Let $X \sim \text{Unif}(0, 1)$. Given $X = x$, the conditional distribution of Y is exponential with parameter x. Find $P(Y > 1)$.

b. Suppose that the probability a certain person gets at least one cold in a year is unknown and modeled as a random variable X whose density is $12x^2(1-x)$ for $0 < x < 1$. Find the probability that over the next 10 years, this person contracts at least one cold in exactly 9 of these years. You should assume that the probability of contracting at least one cold in a year stays the same throughout the 10 years and that each year is independent.

Problem 7. If you liked the problems above or want more practice, our textbook has more great problems. The odd-numbered ones have solutions in the back. Here are some that I recommend (as optional, not to be turned in): 9.1, 9.3, 9.4, 9.8, 9.10, 9.15, 9.19. Feel free to try others, including all the problems in the main sections, which include full explanations.