

§ 4.1-4.2 Expectation

Def Let \bar{X} be a discrete random variable with range $S \subseteq \mathbb{R}$. The probability mass function of \bar{X} is a function $m: S \rightarrow [0,1]$ where $m(k) = P(\bar{X}=k)$ for each $k \in S$. The expectation of \bar{X} (also called the expected value of \bar{X} or mean of \bar{X}) is $E[\bar{X}] = \sum_{k \in S} km(k)$.

Remark ① $E[\bar{X}]$ can be thought of as a weighted average of the possible values of \bar{X} , where the weights are the values given by their probabilities (i.e. their probability mass).

② In analogy with physics we can think of $E[\bar{X}]$ as the center of mass of the distribution.

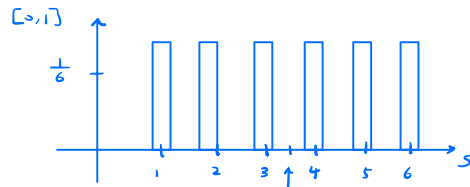
③ Later we will learn that if $\bar{X}_1, \bar{X}_2, \dots$ is an i.i.d. sequence with the same distribution as \bar{X} , then

$$\lim_{n \rightarrow \infty} \frac{\bar{X}_1 + \dots + \bar{X}_n}{n} = E[\bar{X}] \text{ with probability 1.}$$

This is called the Strong Law of Large Numbers.

Example Let $\bar{X} \sim \text{Unif}(\{1,2,3,4,5,6\})$. Find $E[\bar{X}]$.

$$\begin{aligned} E[\bar{X}] &= \sum_{k=1}^6 k \cdot P(\bar{X}=k) = \sum_{k=1}^6 k \cdot \frac{1}{6} \\ &= \frac{1}{6}(1+2+3+4+5+6) = 3.5 \end{aligned}$$



$E[\bar{X}]$ is the center of mass or "balancing point"

Example if $\bar{X} \sim \text{Unif}\{1, 2, \dots, n\}$, then $E[\bar{X}] = \frac{n+1}{2}$

$$\begin{aligned}
 \text{Proof } E[\bar{X}] &= \sum_{k=1}^n k P(\bar{X}=k) \\
 &= \frac{1}{n} \sum_{k=1}^n k \\
 &= \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2} .
 \end{aligned}$$

Example if $\bar{X} \sim \text{Pois}(\lambda)$, then $E[\bar{X}] = \lambda$.

$$\begin{aligned}
 \text{Proof } E[\bar{X}] &= \sum_{k=0}^{\infty} k P(\bar{X}=k) \\
 &= \sum_{k=0}^{\infty} k \cdot e^{-\lambda} \frac{\lambda^k}{k!} \\
 &= \sum_{k=1}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!} \quad (\text{the } k=0 \text{ term was } 0) \\
 &= e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} \quad (\text{cancel with factorial}) \\
 &= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \quad (\text{factor out } \lambda) \\
 &= \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \quad (\text{re-index}) \\
 &= \lambda e^{-\lambda} \cdot e^{\lambda} \quad (\text{Taylor series for } e^x) \\
 &= \lambda \quad (\text{is } \sum_{k=0}^{\infty} \frac{x^k}{k!})
 \end{aligned}$$

Theorem If \bar{X} is a discrete random variable whose range is S and $f: S \rightarrow \mathbb{R}$ is a given function then

$$E[f(\bar{X})] = \sum_{k \in S} f(k) P(\bar{X} = k).$$

Example If the radius of a sphere is $R \sim \text{Unif}(\{1, 2, 3, 4, 5, 6\})$ find the expected value of its volume.

The volume is the random variable $V = \frac{4}{3}\pi R^3$, so

$$\begin{aligned} E[V] &= \sum_{k=1}^6 \frac{4}{3}\pi k^3 \cdot P(\bar{X} = k) \\ &= \frac{4}{3}\pi (1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3) \cdot \frac{1}{6} \\ &= 98\pi \end{aligned}$$

Warning In general, $E[f(\bar{X})] \neq f(E[\bar{X}])$.

However...

Theorem Let $a, b \in \mathbb{R}$ be given constants. Then

$$E[a\bar{X} + b] = a E[\bar{X}] + b.$$

Proof

$$\begin{aligned} E[a\bar{X} + b] &= \sum_{k \in S} (ak + b) P(\bar{X} = k) \\ &= \sum_{k \in S} ak P(\bar{X} = k) + \sum_{k \in S} b P(\bar{X} = k) \\ &= a \sum_{k \in S} k P(\bar{X} = k) + b \sum_{k \in S} P(\bar{X} = k) \\ &= a E[\bar{X}] + b. \end{aligned}$$

Problem 1. Suppose a person always runs a mile in either 4 minutes, 5 minutes, 6 minutes, 10 minutes, or 15 minutes, uniformly at random. Let T denote the time of their run, in minutes, on a given day.

- Find $E[T]$.
- Let τ be the time of their run in hours. Find $E[\tau]$.
- Let S be their speed in miles per hour. Find $E[S]$.

$$\begin{aligned} \textcircled{a} \quad E[T] &= 4\left(\frac{1}{5}\right) + 5\left(\frac{1}{5}\right) + 6\left(\frac{1}{5}\right) + 10\left(\frac{1}{5}\right) + 15\left(\frac{1}{5}\right) \\ &= (4+5+6+10+15)\left(\frac{1}{5}\right) = 8 \text{ minutes} \end{aligned}$$

$$\textcircled{b} \quad \tau = \frac{1}{60}T, \text{ so } E[\tau] = \frac{1}{60}E[T] = \frac{8}{60} = \frac{2}{15} \text{ hours}$$

$$\begin{aligned} \textcircled{c} \quad S &= \frac{1}{T} = \frac{60}{T}, \text{ so } E[S] = \frac{60}{4}\left(\frac{1}{5}\right) + \frac{60}{5}\left(\frac{1}{5}\right) + \frac{60}{6}\left(\frac{1}{5}\right) + \frac{60}{10}\left(\frac{1}{5}\right) + \frac{60}{15}\left(\frac{1}{5}\right) \\ &= (15+12+10+6+4)\left(\frac{1}{5}\right) = 9.4 \text{ mph} \end{aligned}$$

Problem 2. Consider the following gambling game, which costs \$7 to play. You toss a coin 5 times. If the coin comes up heads fewer than 3 times, you lose your money. If the coin comes up heads 3 times you get your money back. The coin comes up heads 4 times, you win \$10. If the coin comes up heads 5 times, you win \$50. Let W represent your net winnings.

- Find the range of W .
- Find the probability mass function of W .
- Find $E[W]$.

$$\textcircled{a} \quad \{-7, 0, 3, 43\}$$

$$\textcircled{b} \quad \text{Let } \bar{X} \sim \text{Bin}\left(5, \frac{1}{2}\right). \text{ Then}$$

$$P(W = -7) = P(\bar{X} \leq 2) = \left(\frac{1}{2}\right)^5 + 5\left(\frac{1}{2}\right)^5 + \binom{5}{2}\left(\frac{1}{2}\right)^5 = 16\left(\frac{1}{2}\right)^5 = \frac{1}{2}$$

$$P(W = 0) = P(\bar{X} = 3) = \binom{5}{3}\left(\frac{1}{2}\right)^5 = \frac{10}{32} = \frac{5}{16}$$

$$P(W = 3) = P(\bar{X} = 4) = \binom{5}{4}\left(\frac{1}{2}\right)^5 = \frac{5}{32}$$

$$P(W = 43) = P(\bar{X} = 5) = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$$

$$\textcircled{c} \quad E[W] = -7\left(\frac{1}{2}\right) + 0\left(\frac{5}{16}\right) + 3\left(\frac{5}{32}\right) + 43\left(\frac{1}{32}\right) = -1.6875$$

Problem 3. Suppose

- $X \sim \text{Unif}\{1, 7\}$. This means $P(X = k) = 1/2$ for $k = 1, 7$.
- $Y \sim \text{Unif}\{1, 2, 3, 4, 5, 6, 7\}$. This means $P(Y = k) = 1/7$ for $k = 1, \dots, 7$.
- $Z = 4$. This means Z is a constant; i.e. $P(Z = 4) = 1$.

Find the following quantities

- $E[X]$
- $E[Y]$
- $E[Z]$
- $E[X^2]$
- $E[Y^2]$
- $E[Z^2]$

$$\textcircled{a} \quad E[X] = 1\left(\frac{1}{2}\right) + 7\left(\frac{1}{2}\right) = 4$$

$$\textcircled{b} \quad E[Y] = \frac{1+7}{2} = 4$$

$$\textcircled{c} \quad E[Z] = 4$$

$$\textcircled{d} \quad E[X^2] = 1\left(\frac{1}{2}\right) + 49\left(\frac{1}{2}\right) = 25$$

$$\textcircled{e} \quad E[Y^2] = (1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2) \left(\frac{1}{7}\right)$$

$$= \frac{7(8)(15)}{6} \left(\frac{1}{7}\right)$$

$$= \frac{4}{3}(15)$$

$$= 20$$

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\textcircled{f} \quad E[Z^2] = 16$$