

§ 4.3, 4.4 Joint distributions, independence

Def Let \bar{X} and \bar{Y} be random variables with respective ranges S and T . Then their joint probability mass function is given by $m(x, y) = P(\bar{X}=x, \bar{Y}=y)$ for each $x \in S, y \in T$.

Example Suppose $m(x, y) = P(\bar{X}=x, \bar{Y}=y) = cxy$ where $x=1, 2$ and $y=1, 2, 3$. Represent m in a table, find c , and find the (marginal) probability mass functions for \bar{X} and \bar{Y} .

	$y=1$	$y=2$	$y=3$	row sums
$x=1$	c	$2c$	$3c$	$6c$
$x=2$	$2c$	$4c$	$6c$	$12c$
col. sums	$3c$	$6c$	$9c$	

table of $m(x,y)$ values

"marginal" probabilities of \bar{X}
 "marginal" probabilities of \bar{Y}

Note that $1 = \sum_{x=1}^2 \sum_{y=1}^3 P(\bar{X}=x, \bar{Y}=y) = 18c$, so $c = \frac{1}{18}$.

And $P(\bar{X}=1) = \sum_{y=1}^3 P(\bar{X}=1, \bar{Y}=y)$ by Law of Total Prob.

$$\begin{aligned}
 &= c + 2c + 3c \\
 &= 6c = \frac{1}{3}
 \end{aligned}$$

Similarly, $P(\bar{X}=2) = 12c = \frac{2}{3}$.

The marginal probability mass function of \bar{X} is

$$P(\bar{X}=k) = \begin{cases} \frac{1}{3} & k=1 \\ \frac{2}{3} & k=2 \end{cases}$$

and the marginal prob. mass function of \bar{Y} is

$$P(\bar{Y}=k) = \begin{cases} \frac{3}{18} & k=1 \\ \frac{6}{18} & k=2 \\ \frac{9}{18} & k=3 \end{cases}$$

Def If \bar{X} and \bar{Y} have joint pmf with ranges S and T respectively, their marginal pmf's

$$\text{are } P(\bar{X}=k) = \sum_{y \in T} P(\bar{X}=k, \bar{Y}=y) \text{ for each } k \in S$$

$$\text{and } P(\bar{Y}=k) = \sum_{x \in S} P(\bar{X}=x, \bar{Y}=k) \text{ for each } k \in T.$$

Example A bag contains 4 red, 2 blue, 3 white balls and we draw a sample of size 2 without replacement

Let R be the number of red balls in the draw, and B the number of blue balls. Find the joint pmf of R and B , their marginal pmfs, $E[RB]$, and $E[R]$ and $E[B]$.

$$P(R=r, B=b) = \frac{\binom{4}{r} \binom{2}{b} \binom{3}{2-(r+b)}}{\binom{9}{2}} \quad \text{where } 0 \leq r+b \leq 2$$

	$b=0$	$b=1$	$b=2$
$r=0$	$\frac{3}{36}$	$\frac{6}{36}$	$\frac{1}{36}$
$r=1$	$\frac{12}{36}$	$\frac{8}{36}$	0
$r=2$	$\frac{6}{36}$	0	0

$$P(R=r) = \begin{cases} \frac{10}{36} & r=0 \\ \frac{20}{36} & r=1 \\ \frac{6}{36} & r=2 \end{cases} \quad P(B=b) = \begin{cases} \frac{21}{36} & b=0 \\ \frac{14}{36} & b=1 \\ \frac{1}{36} & b=2 \end{cases}$$

$$\begin{aligned} E[R+B] &= \sum_{r=0}^2 \sum_{b=0}^2 (r+b) P(R=r, B=b) \\ &= (0+0) P(R=0, B=0) + (0+1) P(R=0, B=1) \\ &\quad + (0+2) P(R=0, B=2) + (1+0) P(R=1, B=0) \\ &\quad + (1+1) P(R=1, B=1) + (2+0) P(R=2, B=0) \\ &= \frac{6}{36} + 2\left(\frac{1}{36}\right) + \frac{12}{36} + 2\left(\frac{3}{36}\right) + 2\left(\frac{6}{36}\right) \\ &= \frac{48}{36} = \frac{4}{3} \end{aligned}$$

$$\begin{aligned} E[RB] &= \sum_{r=0}^2 \sum_{b=0}^2 (rb) P(R=r, B=b) \\ &= \frac{8}{36} \end{aligned}$$

$$\begin{aligned} E[R] &= \sum_{r=0}^2 r P(R=r) = \frac{20}{36} + 2\left(\frac{6}{36}\right) = \frac{32}{36} = \frac{8}{9} \\ E[B] &= \sum_{b=0}^2 b P(B=b) = \frac{14}{36} + 2\left(\frac{1}{36}\right) = \frac{16}{36} = \frac{4}{9} \end{aligned}$$

$$\text{Notice } E[R+B] = E[R] + E[B] \quad \text{but } E[RB] \neq E[R]E[B]$$

Theorem For any random variables \bar{X}, \bar{Y} with ranges S, T

$$\text{respectively } E[g(\bar{X}, \bar{Y})] = \sum_{x \in S} \sum_{y \in T} g(x, y) P(\bar{X}=x, \bar{Y}=y)$$

Theorem If \bar{X} and \bar{Y} are independent, then

$$E[\bar{X}\bar{Y}] = E[\bar{X}]E[\bar{Y}]$$

Theorem For any random variables $E[\bar{X} + \bar{Y}] = E[\bar{X}] + E[\bar{Y}]$.

Proof to come.

Problem 1. Suppose you have the following data on pet ownership in a town with 1000 households.

	Has 0 cats	Has 1 cat	Has 2 cats	total
Has 0 dogs	400	150	150	700
Has 1 dog	50	5	40	95
Has 2 dogs	150	15	40	205
total	600	170	230	1000

Suppose a household is chosen at random, and we let X denote the number of dogs in the household and let Y denote the number of cats in the household.

- Make a table for the joint probability mass function of X and Y .
- Find the marginal probability mass functions of X and Y .
- Find $E[X]$ and $E[Y]$.

(a)

	$y=0$	$y=1$	$y=2$	totals
$x=0$	0.4	0.15	0.15	0.7
$x=1$	0.05	0.005	0.04	0.095
$x=2$	0.15	0.015	0.04	0.205

(b) $P(\bar{X}=x) = \begin{cases} 0.7 & x=0 \\ 0.095 & x=1 \\ 0.205 & x=2 \end{cases}$, $P(\bar{Y}=y) = \begin{cases} 0.6 & y=0 \\ 0.17 & y=1 \\ 0.23 & y=2 \end{cases}$

(c) $E[\bar{X}] = 0.095 + 2(0.205) = 0.505$ $E[\bar{Y}] = 0.17 + 2(0.23) = 0.63$

Problem 2. Suppose we draw two numbers, one at a time without replacement, from the set $\{1, 2, 3, 4\}$. Let X denote the first number drawn, and let Y denote the second number drawn.

- For each value of x and y , compute $P(X = x, Y = y)$ by computing $P(Y = y | X = x)P(X = x)$. Write your answers in a table.
- Find the marginal probability mass functions of X and Y .
- Find $E[X]$ and $E[Y]$.
- Compute $E[XY]$. How does it compare to $E[X]E[Y]$? Is this surprising?
- How does $E[X + Y]$ compare to $E[X] + E[Y]$?

		1	2	3	4
		1	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$
1		$\frac{1}{12}$	0	$\frac{1}{12}$	$\frac{1}{12}$
2		$\frac{1}{12}$	$\frac{1}{12}$	0	$\frac{1}{12}$
3		$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	0
4		$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	0

$$\textcircled{a} \quad P(\bar{X} = x) = \frac{1}{4}, \quad x = 1, 2, 3, 4$$

$$P(\bar{Y} = y) = \frac{1}{4}, \quad y = 1, 2, 3, 4$$

$$\textcircled{b} \quad E[\bar{X}] = 2.5 = E[\bar{Y}]$$

$$\textcircled{c} \quad E[\bar{X}\bar{Y}] = \sum_{x=1}^4 \sum_{y=1}^4 xy P(\bar{X}=x, \bar{Y}=y)$$

$$\begin{aligned} &= \frac{1}{12} (2+3+4 \\ &\quad + 2+6+8 \\ &\quad + 3+6+12 \\ &\quad + 4+8+12) \end{aligned}$$

$$= \frac{70}{12} = 5.833..$$

$E[\bar{X}\bar{Y}] \neq E[\bar{X}]E[\bar{Y}]$ which is not surprising

since \bar{X} and \bar{Y} are not independent

$$\textcircled{d} \quad E[\bar{X} + \bar{Y}] = E[\bar{X}] + E[\bar{Y}] \text{ regardless of lack of independence.}$$