

§ 4.5, 4.6 Linearity of Expectation, Variance

Theorem For any random variables \bar{X} and \bar{Y} ,
 $E[\bar{X} + \bar{Y}] = E[\bar{X}] + E[\bar{Y}]$.

No assumption of independence needed!

Proof

$$\begin{aligned} E[\bar{X} + \bar{Y}] &= \sum_{x \in S} \sum_{y \in T} (x+y) P(\bar{X}=x, \bar{Y}=y) \\ &= \sum_{x \in S} \sum_{y \in T} x P(\bar{X}=x, \bar{Y}=y) + \sum_{x \in S} \sum_{y \in T} y P(\bar{X}=x, \bar{Y}=y) \\ &= \sum_{x \in S} x \sum_{y \in T} P(\bar{X}=x, \bar{Y}=y) + \sum_{y \in T} y \sum_{x \in S} P(\bar{X}=x, \bar{Y}=y) \\ &= \sum_{x \in S} x P(\bar{X}=x) + \sum_{y \in T} y P(\bar{Y}=y) \\ &= E[\bar{X}] + E[\bar{Y}] \end{aligned}$$

Example Let $\bar{X} \sim \text{Bin}(n, p)$. Compute $E[\bar{X}]$.

Note $\bar{X} = \bar{X}_1 + \bar{X}_2 + \dots + \bar{X}_n$ where $\bar{X}_1, \dots, \bar{X}_n \sim \text{Ber}(p)$

Moreover, $E[\bar{X}_i] = p$ for $i=1, \dots, n$. Therefore,

$$E[\bar{X}] = E[\bar{X}_1] + \dots + E[\bar{X}_n] = np.$$

Def The variance of a random variable \bar{X} is given by $V(\bar{X}) = E[(\bar{X} - \mu)^2]$ where $\mu = E[\bar{X}]$.

The standard deviation of \bar{X} is $SD(\bar{X}) = \sqrt{V(\bar{X})}$.

Remarks ① $V(\bar{X}) = E[\bar{X}^2] - \mu^2$.

② Variance measures the "spread" of a distribution.

Example Let $\bar{X} \sim \text{Unif}(\{1, 7\})$, $\bar{Y} \sim \text{Unif}(\{1, 2, 3, 4, 5, 6, 7\})$, $\bar{Z} = 4$.

Compute $V(\bar{X})$, $V(\bar{Y})$, $V(\bar{Z})$.

$$E[\bar{X}] = E[\bar{Y}] = E[\bar{Z}] = 4$$

$$E[\bar{X}^2] = 1^2 \cdot \frac{1}{2} + 7^2 \cdot \frac{1}{2} = 25, \quad E[\bar{Y}^2] = \sum_{k=1}^7 k^2 \cdot \frac{1}{7} = 20, \quad E[\bar{Z}^2] = 16$$

$$V(\bar{X}) = 25 - 4^2 = 9, \quad V(\bar{Y}) = 20 - 4^2 = 4, \quad V(\bar{Z}) = 0.$$

Theorem Let \bar{X}, \bar{Y} be random variables and $a, b \in \mathbb{R}$.

Then ① $V(a\bar{X}) = a^2 V(\bar{X})$

② $V(\bar{X} + b) = V(\bar{X})$

③ $V(\bar{X} + \bar{Y}) = V(\bar{X}) + V(\bar{Y}) + 2(E[\bar{X}\bar{Y}] - \mu_{\bar{X}}\mu_{\bar{Y}})$

where $\mu_{\bar{X}} = E[\bar{X}]$ and $\mu_{\bar{Y}} = E[\bar{Y}]$.

④ if \bar{X} and \bar{Y} are independent,

$$V(\bar{X} + \bar{Y}) = V(\bar{X}) + V(\bar{Y}).$$

Proof of ① $V(a\bar{X}) = E[(a\bar{X})^2] - (E[a\bar{X}])^2$

$$= a^2 E[\bar{X}^2] - a^2 E[\bar{X}]^2$$

$$= a^2 (E[\bar{X}^2] - E[\bar{X}]^2)$$

$$= a^2 V(\bar{X})$$

② $V(\bar{X}+b) = E[(\bar{X}+b)^2] - (E[\bar{X}+b])^2$

$$= E[\bar{X}^2 + 2b\bar{X} + b^2] - (E[\bar{X}] + b)^2$$

$$= E[\bar{X}^2] + 2b E[\bar{X}] + b^2 - (E[\bar{X}]^2 + 2bE[\bar{X}] + b^2)$$

$$= E[\bar{X}^2] - E[\bar{X}]^2 = V(\bar{X}).$$

Example Let $\bar{X} \sim \text{Bin}(n, p)$, Compute $V(\bar{X})$.

Note $\bar{X} = \bar{X}_1 + \dots + \bar{X}_n$ where $\bar{X}_1, \dots, \bar{X}_n \sim \text{Ber}(p)$

are i.i.d. Moreover $E[\bar{X}_i^2] = p$, so

$$V(\bar{X}_i) = E[\bar{X}_i^2] - E[\bar{X}_i]^2 = p - p^2 = p(1-p).$$

Therefore $V(\bar{X}) = V(\bar{X}_1 + \dots + \bar{X}_n)$

$$= V(\bar{X}_1) + \dots + V(\bar{X}_n)$$

$$= np(1-p).$$

Problem 1. Suppose we play a game, which costs \$10 to play, based on the experiment of rolling a die. If our roll is 3 or less, we lose our entry fee. If our rolls is 4, we get our entry fee back. If we roll a 5 or 6, we get our entry fee back along with \$12. Find the expected value, variance, and standard deviation of our net winnings.

Let W denote the winnings. Then

$$P(W=k) = \begin{cases} \frac{1}{2} & k = -10 \\ \frac{1}{6} & k = 0 \\ \frac{1}{3} & k = 12 \end{cases}$$

$$\begin{aligned} E[W] &= (-10)\left(\frac{1}{2}\right) + (0)\left(\frac{1}{6}\right) + (12)\left(\frac{1}{3}\right) \\ &= -5 + 4 = -1 \end{aligned}$$

$$\begin{aligned} E[W^2] &= (-10)^2\left(\frac{1}{2}\right) + (0)^2\left(\frac{1}{6}\right) + (12)^2\left(\frac{1}{3}\right) \\ &= 50 + \frac{144}{3} = \frac{294}{3} \end{aligned}$$

$$V(W) = \frac{294}{3} - (-1)^2 = \frac{291}{3} = 97$$

$$SD(W) = \sqrt{97}$$

Problem 2. Suppose $E[X] = a$, $E[Y] = b$, $E[X^2] = c$, and $E[Y^2] = d$. Express the following in terms of a, b, c, d .

- $E[3X - 4Y]$
- $E[X^2 - 7Y^2 + 3X]$
- $V(-4X + 1)$
- $V(2X - 3Y + 4)$, assuming X and Y are independent

$$V(\bar{X}) = c - a^2, \quad V(\bar{Y}) = d - b^2$$

(a) $3a - 4b$

(b) $c - 7d + 3a$

(c) $16(c - a^2)$

(d) $4(c - a^2) + 9(d - b^2)$

Problem 3. The following small algebra exercises lead up to proving the formula $V(X + Y) = V(X) + V(Y)$ when X and Y are independent.

- Express $E[(X + Y)^2]$ as a sum of three expectations.
- Let $\mu_X = E[X]$ and $\mu_Y = E[Y]$. Express $(\mu_X + \mu_Y)^2$ as a sum of three terms involving μ_X and μ_Y .
- Show that when X and Y are independent, $V(X + Y) = V(X) + V(Y)$.

$$\begin{aligned} \textcircled{a} \quad E[(\bar{X} + \bar{Y})^2] &= E[\bar{X}^2 + 2\bar{X}\bar{Y} + \bar{Y}^2] \\ &= E[\bar{X}^2] + 2E[\bar{X}\bar{Y}] + E[\bar{Y}^2] \quad (*) \end{aligned}$$

$$\textcircled{b} \quad (\mu_{\bar{X}} + \mu_{\bar{Y}})^2 = \mu_{\bar{X}}^2 + 2\mu_{\bar{X}}\mu_{\bar{Y}} + \mu_{\bar{Y}}^2 \quad (+)$$

$$\begin{aligned} \textcircled{c} \quad V(\bar{X} + \bar{Y}) &= E[(\bar{X} + \bar{Y})^2] - E[\bar{X} + \bar{Y}]^2 \\ &= (*) - (+) \\ &= E[\bar{X}^2] - \mu_{\bar{X}}^2 + E[\bar{Y}^2] - \mu_{\bar{Y}}^2 \\ &\quad + 2(E[\bar{X}\bar{Y}] - \mu_{\bar{X}}\mu_{\bar{Y}}) \\ &\quad \quad \quad \underbrace{\hspace{10em}}_{= 0 \text{ when } \bar{X} \text{ and } \bar{Y} \text{ are independent.}} \\ &= V(\bar{X}) + V(\bar{Y}) \end{aligned}$$