§ 4.5, 4.6 Linearity of Expectation, Variance

Theorem For any random variables X and Y,

$E[X+Y] = E[X] + E[Y]$.

No assumption of independence needed!

Proof

$$E[X+Y] = \sum_{x\in S}\sum_{y\in T}(x+y)\,P(X=x, Y=y)$$

$$= \sum_{x\in S}\sum_{y\in T} x\,P(X=x, Y=y) + \sum_{x\in S}\sum_{y\in T} y\,P(X=x, Y=y)$$

$$= \sum_{x\in S} x \sum_{y\in T} P(X=x, Y=y) + \sum_{y\in T}\sum_{x\in S} y\,P(X=x, Y=y)$$

$$= \sum_{x\in S} x\,P(X=x) + \sum_{y\in T} y\,P(Y=y)$$

$$= E[X] + E[Y]$$

Example Let $X \sim Bin(n,p)$. Compute $E[X]$.

Note $X = X_1 + X_2 + \cdots + X_n$ where $X_1, \dots, X_n \sim Ber(p)$

Moreover, $E[X_i] = p$ for $i = 1, \dots, n$. Therefore,

$E[X] = E[X_1] + \cdots + E[X_n] = np$.

Def The <u>variance</u> of a random variable X is given by $V(X) = E[(X-\mu)^2]$ where $\mu = E[X]$.

The <u>standard deviation</u> of X is $SD(X) = \sqrt{V(X)}$.

<u>Remarks</u> ① $V(X) = E[X^2] - \mu^2$.

② Variance measures the "spread" of a distribution.

<u>Example</u> Let $X \sim \text{Unif}(\{1,7\})$, $Y \sim \text{Unif}(\{1,2,3,4,5,6,7\})$, $Z = 4$.

Compute $V(X)$, $V(Y)$, $V(Z)$.

$E[X] = E[Y] = E[Z] = 4$

$E[X^2] = 1^2 \cdot \frac{1}{2} + 7^2 \cdot \frac{1}{2} = 25$, $E[Y^2] = \sum_{k=1}^{7} k^2 \cdot \frac{1}{7} = 20$, $E[Z^2] = 16$

$V(X) = 25 - 4^2 = 9$, $V(Y) = 20 - 4^2 = 4$, $V(Z) = 0$.

<u>Theorem</u> Let X, Y be random variables and $a, b \in \mathbb{R}$.

Then ① $V(aX) = a^2 V(X)$

② $V(X + b) = V(X)$

③ $V(X+Y) = V(X) + V(Y) + 2\left(E[XY] - \mu_X \mu_Y\right)$

where $\mu_X = E[X]$ and $\mu_Y = E[Y]$.

④ if X and Y are independent,

$V(X+Y) = V(X) + V(Y)$.

Proof of ①

$$V(aX) = E[(aX)^2] - (E[aX])^2$$
$$= a^2 E[X^2] - a^2 E[X]^2$$
$$= a^2 (E[X^2] - E[X]^2)$$
$$= a^2 V(X)$$

② $$V(X+b) = E[(X+b)^2] - (E[X+b])^2$$
$$= E[X^2 + 2bX + b^2] - (E[X] + b)^2$$
$$= E[X^2] + 2b E[X] + b^2 - (E[X]^2 + 2bE[X] + b^2)$$
$$= E[X^2] - E[X]^2 = V(X).$$

Example Let $X \sim Bin(n,p)$. Compute $V(X)$.

Note $X = X_1 + \cdots + X_n$ where $X_1, \ldots, X_n \sim Ber(p)$ are i.i.d. Moreover $E[X_i^2] = p$, so

$$V(X_i) = E[X_i^2] - E[X_i]^2 = p - p^2 = p(1-p).$$

Therefore $$V(X) = V(X_1 + \cdots + X_n)$$
$$= V(X_1) + \cdots + V(X_n)$$
$$= np(1-p).$$

Problem 1. Suppose we play a game, which costs $10 to play, based on the experiment of rolling a die. If our roll is 3 or less, we lose our entry fee. If our rolls is 4, we get our entry fee back. If we roll a 5 or 6, we get our entry fee back along with $12. Find the expected value, variance, and standard deviation of our net winnings.

Let W denote the winnings. Then

$$P(W=k) = \begin{cases} \frac{1}{2} & k=-10 \\ \frac{1}{6} & k=0 \\ \frac{1}{3} & k=12 \end{cases}$$

$$E[W] = (-10)\left(\tfrac{1}{2}\right) + (0)\left(\tfrac{1}{6}\right) + (12)\left(\tfrac{1}{3}\right)$$

$$= -5+4 = -1$$

$$E[W^2] = (-10)^2\left(\tfrac{1}{2}\right) + (0)^2\left(\tfrac{1}{6}\right) + (12)^2\left(\tfrac{1}{3}\right)$$

$$= 50 + \frac{144}{3} = \frac{294}{3}$$

$$V(W) = \frac{294}{3} - (-1)^2 = \frac{291}{3} = 97$$

$$SD(W) = \sqrt{97}$$

Problem 2. Suppose $E[X]=a, E[Y]=b, E[X^2]=c$, and $E[Y^2]=d$. Express the following in terms of a,b,c,d.

a. $E[3X-4Y]$

b. $E[X^2-7Y^2+3X]$

c. $V(-4X+1)$

d. $V(2X-3Y+4)$, assuming X and Y are independent

$$V(X) = c-a^2, \quad V(Y) = d-b^2$$

(a) $3a-4b$

(b) $c-7d+3a$

(c) $16(c-a^2)$

(d) $4(c-a^2)+9(d-b^2)$

Problem 3. The following small algebra exercises lead up to proving the formula $V(X+Y) = V(X)+V(Y)$ when X and Y are independent.

a. Express $E\left[(X+Y)^2\right]$ as a sum of three expectations.

b. Let $\mu_X = E[X]$ and $\mu_Y = E[Y]$. Express $(\mu_X+\mu_Y)^2$ as a sum of three terms involving μ_X and μ_Y.

c. Show that when X and Y are independent, $V(X+Y) = V(X)+V(Y)$.

(a) $$E[(X+Y)^2] = E[X^2 + 2XY + Y^2]$$
$$= E[X^2] + 2E[XY] + E[Y^2] \quad (*)$$

(b) $$(\mu_X + \mu_Y)^2 = \mu_X^2 + 2\mu_X\mu_Y + \mu_Y^2 \quad (\dagger)$$

(c) $$V(X+Y) = E[(X+Y)^2] - E[X+Y]^2$$
$$= (*) - (\dagger)$$
$$= E[X^2] - \mu_X^2 + E[Y^2] - \mu_Y^2$$
$$+ 2\big(\underbrace{E[XY] - \mu_X\mu_Y}_{=0 \text{ when } X \text{ and } Y \text{ are independent.}}\big)$$
$$= V(X) + V(Y)$$