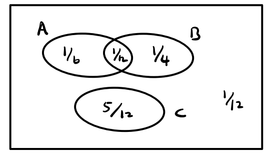
Problem 1. Suppose that A, B, and C are events in an experiment, with C and $A \cup B$ mutually exclusive and

$$P(AB^c) = 1/6, \quad P(BA^c) = 1/4, \quad P(AB) = 1/12, \quad P(C) = 5/12$$



Find the probability of each of the following:

a. A

b. at least one of A or B occurs

c. exactly one of the three events occurs

d. all three events occur

e. at least one of the three events occurs

a) $P(A) = P(AB^c) + P(AB) = \frac{2}{12} + \frac{1}{12} = \frac{1}{4}$

b) $P(A \cup B) = \frac{1}{6} + \frac{1}{12} + \frac{1}{4} = \frac{1}{2}$

c) $P(AB^c \cup BA^c \cup C) = \frac{1}{6} + \frac{1}{4} + \frac{5}{12} = \frac{5}{6}$

d) $P(ABC) = 0$

e) $P(A \cup B \cup C) = \frac{11}{12}$

Problem 2. A symphony orchestra has in its repertoire 30 Haydn pieces, 15 modern pieces, and 9 Beethoven pieces. A program consists of three different pieces from the repertoire. Suppose we choose a program at random. Find the probability that the program has

a. two modern pieces

b. more than one piece of the same type

c. a Hayden piece first, followed by 2 modern pieces

a) $\dfrac{\binom{15}{2}\binom{39}{1}}{\binom{54}{3}}$ b) $1 - \dfrac{\binom{30}{1}\binom{15}{1}\binom{9}{1}}{\binom{54}{3}}$ c) $\dfrac{30 \times 15 \times 14}{54 \times 53 \times 52}$

Problem 3. A bag of Scrabble tiles contains two of each of the letters R, A, N, D, O, and M for a total of 12 tiles. Six tiles are picked without replacement and placed left to right on a Scrabble rack. Find the probability that you:

a. spell R-A-N-D-O-M from left to right

b. pick both R's

c. pick no M's

a) $\dfrac{2^6}{12 \times 11 \times 10 \times 9 \times 8 \times 7}$ b) $\dfrac{\binom{2}{2}\binom{10}{4}}{\binom{12}{6}}$ c) $\dfrac{\binom{10}{6}}{\binom{12}{6}}$

Problem 4. Every Saturday afternoon Carmen plays golf with probability 0.3 or plays squash with probability 0.7. After the golf game, she goes out for a massage with probability 0.55, and after the squash game, she goes out for a massage with probability 0.2.

a. Find the probability that she will go out for a massage.

b. If she goes out for a massage, what is the probability that she played golf?

a) $P(M) = P(M|G)P(G) + P(M|S)P(S)$

$= (0.55)(0.3) + (0.2)(0.7)$

b) $P(G|M) = \dfrac{P(M|G)P(G)}{P(M)} = \dfrac{(0.55)(0.3)}{(0.55)(0.3) + (0.2)(0.7)}$

Problem 5. There are three coins in a box. One is two-headed, one is fair, and one is biased to come up heads with probability 0.75. A coin is selected at random, flipped, and shows heads. What is the probability that it was the two-headed coin?

Let H be the event of flipping heads and let T, F, B be the events of selecting the two-headed coin, the fair coin, and the biased coin with H prob. 0.75.

Then

$$P(T \mid H) = \frac{P(H \mid T)\,P(T)}{P(H \mid T)\,P(T) + P(H \mid F)\,P(F) + P(H \mid B)\,P(B)}$$

$$= \frac{1}{1 + \frac{1}{2} + 0.75}$$

Problem 6. Suppose A, B, C are independent events with respective probabilities 1/6, 1/4, and 1/2. Find the probability that

a. at least one of the events occurs

b. A does not occur, given that both B and C occur

c. A and B occur, given that A or B occur

(a) $P(A \cup B \cup C) = 1 - P((A \cup B \cup C)^c)$

$= 1 - P(A^c B^c C^c) = 1 - P(A^c)P(B^c)P(C^c) = 1 - \left(\frac{5}{6}\right)\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)$

$= 1 - \frac{15}{48} = \frac{33}{48}$

(b) $P(A^c \mid BC) = P(A^c) = \frac{5}{6}$

(c) $P(AB \mid A \cup B) = \frac{P(AB)}{P(A \cup B)} = \frac{P(A)P(B)}{P(A) + P(B) - P(A)P(B)}$

$= \frac{1/24}{1/6 + 1/4 - 1/24} = \frac{1}{9}$

Problem 7. A coin has heads probability 1/3.

a. Find the probability that among 7 tosses of the coin

1. no heads appear
2. exactly 3 heads appear
3. at least 5 heads appear

b. Suppose 5 people each make 7 tosses of the coin. Find the probability that at least 3 of them get no heads.

(a) Let $X \sim \mathrm{Bin}(7, 1/3)$

(1) $P(X = 0) = (2/3)^7$

(2) $P(X = 3) = \binom{7}{3}(1/3)^3(2/3)^4$

(3) $P(X \geq 5) = \binom{7}{5}(1/3)^5(2/3)^2 + \binom{7}{6}(1/3)^6(2/3) + (1/3)^7$

(b) Let $Y \sim \mathrm{Bin}(5, p)$ where $p = (2/3)^7$. Then

$P(Y \geq 3) = \binom{5}{3} p^3 (1-p)^2 + \binom{5}{4} p^4 (1-p) + p^5$

Problem 8. The number of times that a person contracts a cold in a given year is Poisson distributed. The probability they contract no colds in a year is 30%. Find the probability that over the next 2 years, the person contracts exactly 5 colds.

Let $X \sim Pois(\lambda)$ be the number of colds a person gets over one year and $Y \sim Pois(2\lambda)$ over 2 years.

Then $P(X=0) = 0.3 \Rightarrow e^{-\lambda} = 0.3 \Rightarrow -\lambda = \ln(0.3)$.

So $P(Y=5) = e^{2\ln(0.3)} \frac{(-2\ln 0.3)^5}{5!}$

Problem 9. A hotel has 25 single rooms, all occupied, and numbered 1 to 25. Each room, independent of the others, makes on average 3 phone calls per hour.

a. Give an appropriate distribution for the number of phone calls made by a single room in a one hour period, including any relevant parameters.

b. Give an appropriate distribution for the number of rooms which each made exactly 2 phone calls in a hone hour period, including any relevant parameters.

c. Find the probability that

1. at least 1 phone call was made by room 5 in a one hour period.
2. at least 2 phone calls were made in total by rooms 5 and 6 in a one hour period.
3. at least 3 rooms each made exactly 2 phone calls on a one hour period.

(a) $Pois(3)$

(b) Let $X \sim Pois(3)$ and let $p = P(X=2) = e^{-3}\frac{3^2}{2!} = 4.5e^{-3}$

Then $Y \sim Bin(25, p)$ is the appropriate distribution.

(c) ① $P(X \geq 1) = 1 - P(X=0) = 1 - e^{-3}$

② Let $X_1, X_2 \sim Pois(3)$ be i.i.d. Then

$$P(X_1 + X_2 \geq 2) = 1 - P(X_1 + X_2 \leq 1)$$

$$= 1 - \left(P(X_1=0, X_2=0) + P(X_1=0, X_2=1) + P(X_1=1, X_2=0)\right)$$

$$= 1 - \left((e^{-3})^2 + 2(e^{-3})(e^{-3}\cdot 3)\right)$$

$$= 1 - 7e^{-6}$$

③ $P(Y \geq 3) = 1 - P(Y \leq 2) = 1 - (1-p)^{25} + 25p(1-p)^{24} + \binom{25}{2}p^2(1-p)^{23}$

Problem 10. Consider a random variable X whose probability mass function is given below. Find the following quantities.

$$P(X = x) = \begin{cases} c & x = -3 \\ 2c & x = -2 \\ 2c & x = 0 \\ 3c & x = 1 \\ 4c & x = 2. \end{cases}$$

a. c

b. $P(X > -2)$

c. $P(-2 \leq X < 2)$

d. $E[X]$

e. $E[X^2]$

a) $12c = 1 \Rightarrow c = 1/12$

b) $P(X > -2) = 9c = 3/4$

c) $P(-2 \leq X < 2) = 7c = 7/12$

d) $E[X] = -3c - 2(2c) + 3c + 2(4c)$

$= 4c = 1/3$

e) $E[X^2] = 9c + 4(2c) + 3c + 4(4c)$

$= 36c = 3.$