Problem 1. Suppose that A, B, and C are events in an experiment, with C and $A \cup B$ mutually exclusive and

$$P(AB^c) = 1/6, \quad P(BA^c) = 1/4, \quad P(AB) = 1/12, \quad P(C) = 5/12$$

Find the probability of each of the following:

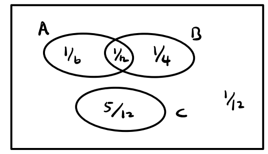
a. A

b. at least one of A or B occurs

c. exactly one of the three events occurs

d. all three events occur

e. at least one of the three events occurs



(a) $P(A) = P(AB^c) + P(AB) = \frac{2}{12} + \frac{1}{12} = \frac{1}{4}$

(b) $P(A \cup B) = \frac{1}{6} + \frac{1}{12} + \frac{1}{4} = \frac{1}{2}$

(c) $P(AB^c \cup BA^c \cup C) = \frac{1}{6} + \frac{1}{4} + \frac{5}{12} = \frac{5}{6}$

(d) $P(ABC) = 0$

(e) $P(A \cup B \cup C) = \frac{11}{12}$

Problem 2. A symphony orchestra has in its repertoire 30 Haydn pieces, 15 modern pieces, and 9 Beethoven pieces. A program consists of three different pieces from the repertoire. Suppose we choose a program at random. Find the probability that the program has

a. two modern pieces

b. more than one piece of the same type

c. a Hayden piece first, followed by 2 modern pieces

(a) $\dfrac{\binom{15}{2}\binom{39}{1}}{\binom{54}{3}}$ (b) $1 - \dfrac{\binom{30}{1}\binom{15}{1}\binom{9}{1}}{\binom{54}{3}}$ (c) $\dfrac{30 \times 15 \times 14}{54 \times 53 \times 52}$

Problem 3. A bag of Scrabble tiles contains two of each of the letters R, A, N, D, O, and M for a total of 12 tiles. Six tiles are picked without replacement and placed left to right on a Scrabble rack. Find the probability that you:

a. spell R-A-N-D-O-M from left to right

b. pick both R's

c. pick no M's

(a) $\dfrac{2^6}{12 \times 11 \times 10 \times 9 \times 8 \times 7}$ (b) $\dfrac{\binom{2}{2}\binom{10}{4}}{\binom{12}{6}}$ (c) $\dfrac{\binom{10}{6}}{\binom{12}{6}}$

Problem 4. Every Saturday afternoon Carmen plays golf with probability 0.3 or plays squash with probability 0.7. After the golf game, she goes out for a massage with probability 0.55, and after the squash game, she goes out for a massage with probability 0.2.

a. Find the probability that she will go out for a massage.

b. If she goes out for a massage, what is the probability that she played golf?

(a) $P(M) = P(M|G)P(G) + P(M|S)P(S)$

$= (0.55)(0.3) + (0.2)(0.7)$

(b) $P(G|M) = \dfrac{P(M|G)P(G)}{P(M)} = \dfrac{(0.55)(0.3)}{(0.55)(0.3) + (0.2)(0.7)}$

Problem 5. There are three coins in a box. One is two-headed, one is fair, and one is biased to come up heads with probability 0.75. A coin is selected at random, flipped, and shows heads. What is the probability that it was the two-headed coin?

Let H be the event of flipping heads and let T, F, B be the events of selecting the two-headed coin, the fair coin, and the biased coin with H prob. 0.75.

Then

$$P(T\mid H) = \frac{P(H\mid T)\,P(T)}{P(H\mid T)\,P(T) + P(H\mid F)\,P(F) + P(H\mid B)\,P(B)}$$

$$= \frac{1}{1 + \tfrac{1}{2} + 0.75}$$

Problem 6. Suppose A, B, C are independent events with respective probabilities 1/6, 1/4, and 1/2. Find the probability that

a. at least one of the events occurs

b. A does not occur, given that both B and C occur

c. A and B occur, given that A or B occur

(a) $P(A \cup B \cup C) = 1 - P((A \cup B \cup C)^c)$

$= 1 - P(A^c B^c C^c) = 1 - P(A^c)P(B^c)P(C^c) = 1 - \left(\tfrac{5}{6}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{1}{2}\right)$

$= 1 - \frac{15}{48} = \frac{33}{48}$

(b) $P(A^c \mid BC) = P(A^c) = \frac{5}{6}$

(c) $P(AB \mid A \cup B) = \dfrac{P(AB)}{P(A\cup B)} = \dfrac{P(A)P(B)}{P(A) + P(B) - P(A)P(B)}$

$= \dfrac{1/24}{1/6 + 1/4 - 1/24} = \dfrac{1}{9}$

Problem 7. A coin has heads probability 1/3.

a. Find the probability that among 7 tosses of the coin

1. no heads appear
2. exactly 3 heads appear
3. at least 5 heads appear

b. Suppose 5 people each make 7 tosses of the coin. Find the probability that at least 3 of them get no heads.

(a) Let $X \sim Bin(7, 1/3)$

(1) $P(X = 0) = (2/3)^7$

(2) $P(X = 3) = \binom{7}{3}(1/3)^3(2/3)^4$

(3) $P(X \geq 5) = \binom{7}{5}(1/3)^5(2/3)^2 + \binom{7}{6}(1/3)^6(2/3) + (1/3)^7$

(b) Let $Y \sim Bin(5, p)$ where $p = (2/3)^7$. Then

$P(Y \geq 3) = \binom{5}{3}p^3(1-p)^2 + \binom{5}{4}p^4(1-p) + p^5$

Problem 8. For the following situations determine whether the binomial distribution is a reasonable model for the given random variable. If so, state its parameters. If not, explain why.

a. Grant believes there is a 40 percent chance of rain tomorrow. Let X indicate the presence or absence of rain tomorrow.

b. Dana and Curtis are playing a strategy game. They are equally likely to win, and play 10 matches. Let X denote the number of Dana's wins out of those matches.

c. Eddie spends his free afternoons watching ship traffic in the harbor. Each hour about 4 large ships arrive to dock at the port. Let X be the number of large ships which arrive in the next hour.

d. Marilyn is playing a board game where income per turn is generated by rolling a standard six-sided die and multiplying the result by 100. Let X be the income earned on a turn.

(a) $\text{Bin}(1, 0.4)$

(b) $\text{Bin}(10, 1/2)$

(c) no, no fixed number of trials

(d) no, not a success/failure experiment

Problem 9. Let $X_1 \sim \text{Ber}(0.2)$ and $X_2 \sim \text{Ber}(0.7)$ be independent random variables. Find the probability mass function of $Y = X_1 + X_2$. What is the probability mass function of Y if X_1 and X_2 have the same parameter p?

$$P(Y=0) = P(X_1=0, X_2=0) = (0.8)(0.3) = 0.24$$

$$P(Y=1) = P(X_1=1, X_2=0) + P(X_1=0, X_2=1) = (0.2)(0.3) + (0.8)(0.7) = 0.62$$

$$P(Y=2) = P(X_1=1, X_2=1) = (0.2)(0.7) = 0.14$$

$$P(Y=k) = \begin{cases} 0.24 & k=0 \\ 0.62 & k=1 \\ 0.14 & k=2 \end{cases}$$

If $X_1, X_2 \sim \text{Ber}(p)$, then $Y \sim \text{Bin}(2,p)$ and so

$$P(Y=k) = \binom{2}{k} p^k (1-p)^{2-k} \quad \text{for } k = 0, 1, 2.$$