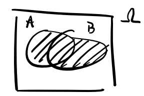
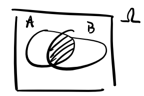
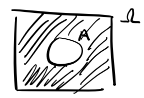
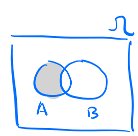
§ 1.4 Properties of probabilities

Let $A, B \subseteq \Omega$ be events. Here are some operations:

operation	notation	description	
union	$A \cup B$	A or B (or both)	
intersection	$A \cap B$ (or AB)	A and B (simultaneously)	
complement	A^c	not A	
implication	$A \subseteq B$	A implies B (or if A then B)	

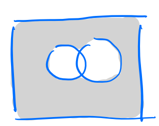
<u>Example</u> Express each of the following using set operations and draw the corresponding Venn diagram

① A but not B occurs



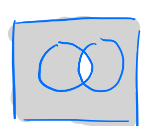
$A \cap B^c = AB^c$

② neither A nor B occurs



$(A \cup B)^c$ see worksheet for another way

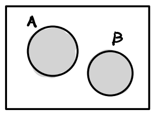
③ at most one of A or B occurs



$(AB)^c$ see worksheet for another way

Def Events $A, B \subseteq \Omega$ are called *mutually exclusive* if $AB = \phi$ (ie. they are disjoint or in other words have no outcomes in common).

"empty set"



Addition Rule If $A, B \subseteq \Omega$ are mutually exclusive then $P(A \cup B) = P(A) + P(B)$.

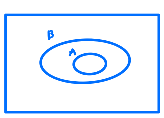
Theorem The following properties hold for any events $A, B \subseteq \Omega$:

① if $A \subseteq B$, then $P(A) \leq P(B)$

② $P(A^c) = 1 - P(A)$

③ $P(A \cup B) = P(A) + P(B) - P(AB)$

Proof ① Observe that $B = A \cup BA^c$ and A and BA^c are mutually exclusive.



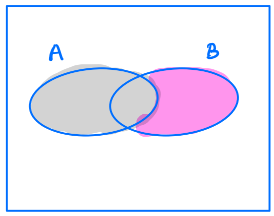
Therefore $P(B) = P(A) + P(BA^c)$

$\geq P(A) + 0 = P(A)$.

② Observe that $\Omega = A \cup A^c$ and A and A^c are mutually exclusive events. Therefore

$1 = P(\Omega) = P(A) + P(A^c)$, which implies $P(A) = 1 - P(A^c)$.

③



Observe that

$A \cup B = A \cup BA^c$ and

A and BA^c are mutually exclusive

Therefore $P(A \cup B) = P(A) + P(BA^c)$.

Moreover, $B = BA^c \cup AB$ and BA^c and AB are mutually exclusive events. Therefore

$$P(B) = P(BA^c) + P(AB)$$

which implies $P(BA^c) = P(B) - P(AB)$. Thus

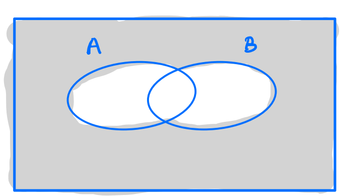
$$\begin{aligned} P(A \cup B) &= P(A) + P(BA^c) \\ &= P(A) + P(B) - P(AB). \end{aligned}$$

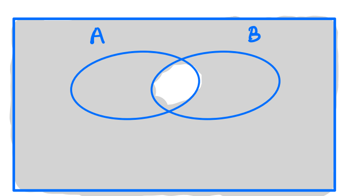
Problem 1. Let A, B, C be events. Draw Venn diagrams for the following events and make note of the formulas that you seem to be deriving.

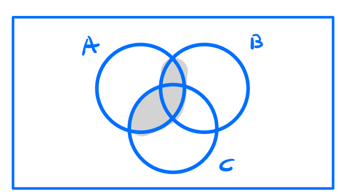
1. $(A \cup B)^c$
2. $A^c B^c$
3. $(AB)^c$
4. $A^c \cup B^c$
5. $A(B \cup C)$
6. $AB \cup AC$

Note for after class. The formulas you derived show that set operations have *distributive* properties. You've derived three formulas but there's a fourth. What is it? Put all four in your notes!

$A \cup (BC) = (A \cup B)(A \cup C)$







$(A \cup B)^c$

$= A^c B^c$

$(AB)^c$

$= A^c \cup B^c$

$A(B \cup C)$

$= AB \cup AC$

Problem 2. Suppose A and B are two given events. For each of the following new events, draw a Venn diagram and then express it in terms of A and B using intersection, union, complement, or some combination of these operations.

a. At least one of the two events occurs,

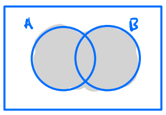
b. Both of the events occur,

c. Neither event occurs,

d. Exactly one of the two events occur.

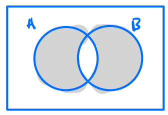
e. At most one of the two events occurs.

(a)



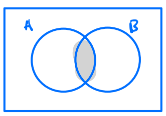
$A \cup B$

(d)



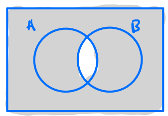
$AB^c \cup BA^c$

(b)



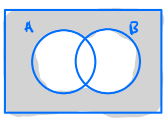
AB

(e)



$(AB)^c$

(c)



$(A \cup B)^c$

Problem 3. Consider the events A and B from the previous problem and suppose that $P(A) = 0.4$, $P(B) = 0.5$, and $P(A \cup B) = 0.8$. Find the probabilities of each of the five new events of the previous problem.

$$P(AB) = P(A) + P(B) - P(A \cup B) = 0.1$$

(a) $P(A \cup B) = 0.8$

(b) $P(AB) = 0.1$

(c) $P((A \cup B)^c) = 1 - 0.8 = 0.2$

(d) $P(AB^c \cup BA^c) = 0.8 - 0.1 = 0.7$

(e) $P((AB)^c) = 1 - P(AB) = 0.9$

Problem 4. Suppose $P(A \cup B) = 0.6$ and $P(A \cup B^c) = 0.8$. Find $P(A)$.

$$P(A \cup B^c) = P(A) + P(B^c) - P(AB^c)$$

$$+\ P(A \cup B) = P(A) + P(B) - P(AB)$$

$$1.4 = 2P(A) + 1 - \underbrace{(P(AB^c) + P(AB))}_{=P(A)}$$

$$0.4 = P(A)$$

Problem 5. Zahkeyah is taking two books along on her holiday vacation. With probability 0.5, she will like the first book; with probability 0.4, she will like the second book; and with probability 0.3, she will like both books. Find the probability that she likes neither book. *Make sure to clearly define some notation using sentences and then show your steps.*

Let A be the event she likes the first book and let B be the event she likes the second one.

$$P((A \cup B)^c) = 1 - P(A \cup B)$$

$$= 1 - P(A) - P(B) + P(AB)$$

$$= 1 - 0.5 - 0.4 + 0.3 = 0.4$$

Problem 6. The probability that a visit to a Primary Care Physician (PCP)'s office results in neither lab work nor a referral to a specialist is 35%. On the other hand, 30% of PCP visits are referred to a specialist and 40% require lab work. Find the probability that a PCP visit results in both lab work and a referral to a specialist. *Make sure to clearly define some notation using sentences and then show your steps.*

Let L be the event the visit results in lab work, let S be the event it results in a specialist referral.

$$P(LS) = P(L) + P(S) - P(L \cup S)$$

$$= 40\% + 30\% - (1 - 35\%)$$

$$= 70\% - 65\% = 5\%.$$