Integration review

$\int_a^b f(x)\,dx$ represents area between $y=f(x)$ and $y=0$ over the interval $[a,b]$

An antiderivative of $f(x)$ is a function $F(x)$ such that $F'(x)=f(x)$

Common antiderivatives

$$\int x\,dx = \frac{1}{2}x^2 + C \qquad \int x^{-2}\,dx = -x^{-1} + C$$

$$\int x^2\,dx = \frac{1}{3}x^3 + C \qquad \int x^{-3}\,dx = -\frac{1}{2}x^{-2} + C$$

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \quad \text{when } n \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

Fundamental Theorem of Calculus (part II)

If $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(t)\,dt = F(t)\Big|_a^b = F(b) - F(a).$$

Fundamental Theorem of Calculus (part I)

The function $F(x) = \int_a^x f(t)\,dt$ is an antiderivative of $f(x)$. That is, $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

u-substitution

$$\int_0^3 e^{5x}\,dx = \int_0^{15} \tfrac{1}{5} e^u\,du \qquad u = 5x$$

$$\qquad du = 5dx$$

$$= \tfrac{1}{5} e^u \Big|_0^{15} \qquad \tfrac{1}{5}du = dx$$

$$= \tfrac{1}{5}(e^{15} - 1)$$

$$\int_0^2 x e^{x^2}\,dx = \int_0^4 \tfrac{1}{2} e^u\,du \qquad u = x^2$$

$$du = 2x\,dx$$

$$= \tfrac{1}{2} e^u \Big|_0^4 \qquad \tfrac{1}{2}du = x\,dx$$

$$= \tfrac{1}{2}(e^4 - 1)$$

improper integrals

(1) turn into limit, (2) compute integral inside limit (3) take limit

$$\int_0^\infty e^{-3x}\,dx = \lim_{b\to\infty} \int_0^b e^{-3x}\,dx \qquad u = -3x$$

$$du = -3dx$$

$$= \lim_{b\to\infty} \frac{-1}{3} \int_0^{-3b} e^u\,du$$

$$= \lim_{b\to\infty} \frac{-1}{3} e^u \Big|_0^{-3b} = \lim_{b\to\infty} \frac{-1}{3}(e^{-3b} - 1) = \frac{1}{3}$$

$$\int_1^\infty x^{-3}\,dx = \lim_{b\to\infty} \int_1^b x^{-3}\,dx = \lim_{b\to\infty} -\frac{1}{2} x^{-2} \Big|_1^b$$

$$= \lim_{b\to\infty} -\frac{1}{2}(b^{-2} - 1) = \frac{1}{2}$$

§ 6.1 Probability density functions

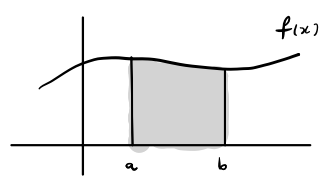
Def A random variable X is a <u>continuous random variable</u> if its range $S \subseteq \mathbb{R}$ is uncountable (eg. an interval).

Def Let X be a continuous random variable. Then X has <u>probability density function</u> (or just <u>density</u>) f if

(1) $f(x) \geq 0$ for all $x \in \mathbb{R}$

(2) $\displaystyle\int_{-\infty}^{\infty} f(t)\,dt = 1$

(3) for any $A \subseteq \mathbb{R}$, $P(X \in A) = \displaystyle\int_A f(t)\,dt$



$$P(a < X < b) = \int_a^b f(t)\,dt$$

Remarks

① if $A = (a, b]$, we write $P(X \in A)$ instead as

$$P(a < X \leq b) = \int_a^b f(t)\,dt$$

② if $A = (a, \infty)$, we write $P(X \in A)$ instead as

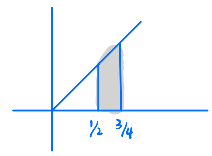
$$P(X > a) = \int_a^{\infty} f(t)\,dt$$

③ if $A = \{a\}$, then $P(X \in A) = \displaystyle\int_a^a f(t)\,dt = 0$

④ $P(X \leq a) = P(X < a)$ by ③.

Example Let X be a continuous random variable with density $f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$

Find $P(\frac{1}{2} < X \le \frac{3}{4})$, $P(X > \frac{1}{2})$



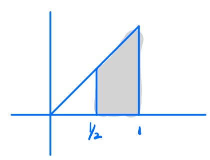
$$P(\tfrac{1}{2} < X \le \tfrac{3}{4}) = \int_{1/2}^{3/4} f(t)\,dt$$

$$= \int_{1/2}^{3/4} 2t\,dt$$

$$= t^2 \Big|_{1/2}^{3/4}$$

$$= (3/4)^2 - (1/2)^2$$

$$= 9/16 - 4/16 = 5/16$$



$$P(X > \tfrac{1}{2}) = \int_{1/2}^{\infty} f(t)\,dt$$

$$= \int_{1/2}^{1} 2t\,dt$$

$$= t^2 \Big|_{1/2}^{1}$$

$$= 1 - (1/2)^2 = 3/4$$

Example Suppose X has density $f(x) = \begin{cases} 2xe^{-x^2} & x > 0 \\ 0 & x \leq 0 \end{cases}$

Find $P(X > 1)$

$$= \int_1^\infty f(t)\,dt$$

$$= \int_1^\infty 2te^{-t^2}\,dt$$

$$= \lim_{b\to\infty} \int_1^b 2te^{-t^2}\,dt \qquad u = -t^2$$

$$du = -2t\,dt$$

$$= \lim_{b\to\infty} \int_{-1}^{-b^2} -e^u\,du$$

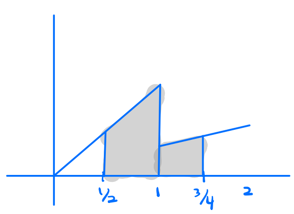
$$= \lim_{b\to\infty} e^u \Big|_{-b^2}^{-1}$$

$$= \lim_{b\to\infty} \left(e^{-1} - e^{-b^2}\right)$$

$$= e^{-1}$$

Example Suppose X has density $f(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ \frac{1}{3}x & \text{if } 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$

Find $P(\frac{1}{2} < X < 3/2)$



$$\int_{1/2}^{3/2} f(t)\,dt = \int_{1/2}^{1} t\,dt + \int_1^{3/2} \frac{1}{3}t\,dt$$

$$= \frac{1}{2}t^2\Big|_{1/2}^{1} + \frac{1}{6}t^2\Big|_1^{3/2}$$

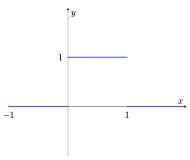
$$= \frac{1}{2}\left(1 - \frac{1}{4}\right) + \frac{1}{6}\left(\frac{9}{4} - 1\right)$$

$$= \frac{3}{8} + \frac{5}{24} = \frac{14}{24} = \frac{7}{12}$$

Problem 1. Consider the piecewise defined function $f : \mathbb{R} \to \mathbb{R}$ given by.

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Its plot is given here:



Compute each of the following using just the elementary fact that the area of a rectangle is bh where b denotes the length of the base and h denotes the length of the height.

a. $\int_0^{1/4} f(t)\,dt$
b. $\int_0^{1/2} f(t)\,dt$
c. $\int_0^{2/3} f(t)\,dt$
d. $\int_{1/3}^{2/3} f(t)\,dt$
e. $\int_2^{3} f(t)\,dt$
f. $\int_{1/2}^{2} f(t)\,dt$
g. $\int_{-\infty}^{0} f(t)\,dt$
h. $\int_{-\infty}^{1} f(t)\,dt$
i. $\int_{-\infty}^{\infty} f(t)\,dt$
j. $\int_0^{x} f(t)\,dt$ if $0 < x < 1$ (your answer will be in terms of x)
k. $\int_0^{x} f(t)\,dt$ if $x \geq 1$
l. $\int_0^{x} f(t)\,dt$ if $x \leq 0$

a) $1/4$
b) $1/2$
c) $2/3$
d) $1/3$
e) 0
f) $1/2$
g) 0
h) 1
i) 1
j) x
k) 1
l) 0

Problem 2. Let X be a random variable with density f given by

$$f(x) = \begin{cases} cx^2 & -2 \leq x \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

a. What value of c makes it so that $\int_{-\infty}^{\infty} f(t)\,dt = 1$?

b. For each of the following definite integrals, draw a plot of $f(x)$, shade in the area represented by the integral, and then compute a value for the integral/area.

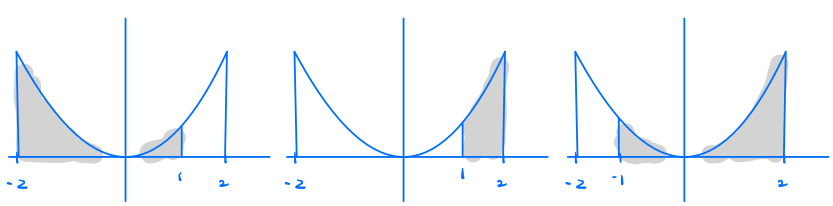
1. $\int_{-\infty}^{1} f(t)\,dt$
2. $\int_1^{\infty} f(t)\,dt$ (can you use your answer to the previous integral when computing this?)
3. $\int_{-1}^{2} f(t)\,dt$

c. What probabilities do the previous integrals represent?

a) $$1 = \int_{-\infty}^{\infty} f(t)\,dt = \int_{-2}^{2} ct^2\,dt = 2c\int_0^2 t^2\,dt = \frac{2c}{3}t^3\Big|_0^2 = \frac{16c}{3}$$

$$C = \frac{3}{16}$$

b)



1) $\int_{-\infty}^{1} f(t)\,dt$

$$= \frac{1}{2} + \int_0^1 cx^2\,dx = \frac{1}{2} + \frac{c}{3}x^3\Big|_0^1 = \frac{1}{2} + \frac{1}{16} = \frac{9}{16}$$

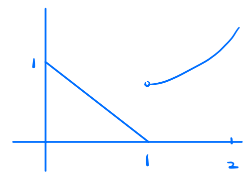
2) $\int_1^{\infty} f(t)\,dt = 1 - \frac{9}{16} = \frac{7}{16}$

3) $\int_{-1}^{2} f(t)\,dt = \frac{9}{16}$

c) $P(X \leq 1)$, $P(X \geq 1)$, $P(-1 \leq X \leq 2)$

Problem 3. Let X be a random variable with density f given by

$$f(x) = \begin{cases} 1-x & 0 \le x < 1 \\ cx^2 & 1 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$



a. What value of c makes it so that $\int_{-\infty}^{\infty} f(t)\,dt = 1$?

b. For each of the following definite integrals, draw a plot of $f(x)$, shade in the area represented by the integral, and then compute a value for the integral/area.

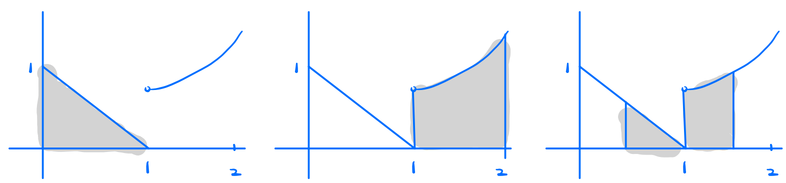
1. $\int_{-\infty}^{1} f(t)\,dt$
2. $\int_{1}^{\infty} f(t)\,dt$ (can you use your answer to the previous integral when computing this?)
3. $\int_{1/2}^{3/2} f(t)\,dt$

c. What probabilities do the previous integrals represent?

ⓐ $1 = \int_{-\infty}^{\infty} f(t)\,dt = \int_0^1 (1-t)\,dt + \int_1^2 ct^2\,dt = \frac{1}{2} + \frac{c}{3}t^3 \Big|_1^2 = \frac{1}{2} + \frac{7c}{3}$

$\Rightarrow \frac{7c}{3} = \frac{1}{2} \Rightarrow c = \frac{3}{14}$

ⓑ



① $\int_{-\infty}^{1} f(t)\,dt = \frac{1}{2}$

② $\int_{1}^{\infty} f(t)\,dt = \frac{1}{2}$

③ $\int_{1/2}^{3/2} f(t)\,dt = \int_{1/2}^{1} (1-t)\,dt + \int_1^{3/2} ct^2\,dt$

$= \frac{1}{8} + \frac{c}{3}t^3 \Big|_1^{3/2} = \frac{1}{8} + \frac{1}{14}\left((3/2)^3 - 1\right)$

$= \frac{1}{8} + \frac{1}{14}\left(\frac{27-8}{8}\right)$

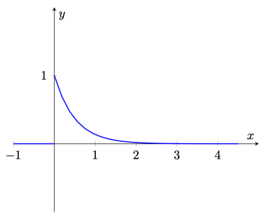
$= \frac{14}{112} + \frac{9}{112} = \frac{23}{112}$

ⓒ $P(X \le 1)$, $P(X \ge 1)$, $P(\frac{1}{2} \le X \le \frac{3}{2})$

Problem 4. Let X be a random variable whose density is given by

$$f(x) = \begin{cases} ce^{-2x} & x \geq 0 \\ 0 & x < 0. \end{cases}$$

The plot of f is given here:



a. Find c so that $\int_{-\infty}^{\infty} f(t)\,dt = 1$

b. Compute:

1. $P(X < 1)$
2. $P(X = 1)$
3. $P(1 < X < 2)$
4. $P(X > 2)$
5. $P(X \leq x)$ for an arbitrary positive number x
6. $P(X \leq x)$ for an arbitrary negative number x

(a) $$1 = \int_0^\infty ce^{-2x}\,dx$$

$$= \lim_{b\to\infty} \int_0^b ce^{-2x}\,dx \qquad u = -2x, \quad du = -2dx, \quad -\tfrac{1}{2}du = dx$$

$$= -\frac{c}{2} \lim_{b\to\infty} \int_0^{-2b} e^u\,du$$

$$= -\frac{c}{2} \lim_{b\to\infty} e^u \Big|_0^{-2b}$$

$$= \frac{c}{2} \lim_{b\to\infty} (1 - e^{-2b})$$

$$= \frac{c}{2} \implies c = 2$$

(b) ① $$P(X<1) = \int_0^1 2e^{-2t}\,dt$$

$$= -e^{-2t}\Big|_0^1$$

$$= 1 - e^{-2}$$

② $P(X = 1) = 0$

③ $$P(1 < X < 2) = \int_1^2 2e^{-2t}\,dt$$

$$= -e^{-2t}\Big|_1^2$$

$$= e^{-2} - e^{-4}$$

④ $$P(X > 2) = 1 - P(X \leq 2)$$

$$= 1 - \big(P(X<1) + P(1<X<2)\big)$$

$$= 1 - (1 - e^{-2} + e^{-2} - e^{-4})$$

$$= e^{-4}$$

⑤ $$P(X \leq x) = \int_0^x 2e^{-2t}\,dt = -e^{-2t}\Big|_0^x$$

$$= 1 - e^{-2x}$$

⑥ $P(X \leq x) = 0.$