§ 6.2 Cumulative distribution functions

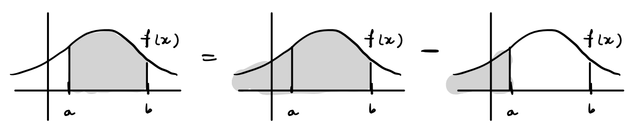
Def Let X be a random variable. The cumulative distribution function $F : \mathbb{R} \to [0,1]$ of X is given by

$$F(x) = P(X \le x)$$

Remarks ① When X has density f, $F(x) = \int_{-\infty}^{x} f(t)\,dt$.

Therefore $F'(x) = f(x)$ by the FTOC, part I

② If X has cdf F, $P(a \le X \le b) = F(b) - F(a)$



$P(a \le X \le b) \quad = \quad F(b) \quad - \quad F(a)$

Example Suppose the amount of time you wait until the next bus arrives at the stone shelter is a random variable X (in minutes) with CDF $F(x) = \begin{cases} 1 - e^{-x/30} & x > 0 \\ 0 & x \le 0 \end{cases}$

Find the probability you wait

① no more than 10 minutes $P(X \le 10) = F(10) = 1 - e^{-10/30}$
$= 1 - e^{-1/3}$

② at least 10 minutes $P(X \ge 10) = 1 - P(X < 10) = 1 - P(X \le 10)$
$= 1 - F(10)$
$= e^{-1/3}$

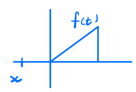
③ between 10 and 15 minutes.

$$P(10 \le X \le 15) = F(15) - F(10)$$
$$= (1 - e^{-15/30}) - (1 - e^{-10/30})$$
$$= e^{-1/3} - e^{-1/2}$$

Example Suppose X has density $f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$

① Find the CDF of X.

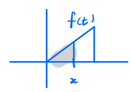
Case I Suppose $x < 0$. Then



$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$= \int_{-\infty}^{x} 0\,dt = 0$$

Case II Suppose $0 \le x \le 1$. Then



$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$= \int_{-\infty}^{0} f(t)\,dt + \int_{0}^{x} f(t)\,dt$$

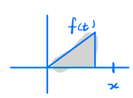
$$= \int_{-\infty}^{0} 0\,dt + \int_{0}^{x} 2t\,dt$$

$$= \int_{0}^{x} 2t\,dt$$

$$= t^2 \Big|_{0}^{x}$$

$$= x^2$$

Case III Suppose $x > 1$. Then



$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$= \int_{-\infty}^{0} f(t)\,dt + \int_{0}^{1} f(t)\,dt + \int_{1}^{x} f(t)\,dt$$

$$= \int_{-\infty}^{0} 0\,dt + \int_{0}^{1} 2t\,dt + \int_{1}^{x} 0\,dt$$

$$= 1$$

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

① Find $P(\frac{1}{2} < X < \frac{3}{2})$

$$= F(3/2) - F(1/2)$$

$$= 1 - (1/2)^2 = 1 - \frac{1}{4} = \frac{3}{4}.$$

Problem 1. Suppose X is a continuous random variable with density

$$f(x) = \begin{cases} c & -4 \le x \le 7 \\ 0 & \text{otherwise.} \end{cases}$$

a. Draw the graph of f and find c.

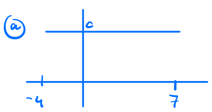
b. Find an expression for $F(x) = P(X \le x)$ in terms of x when

1. $x < -4$
2. $-4 \le x \le 7$
3. $x > 7$

c. Use $F(x)$ to find the following probabilities. Do not do any integration.

1. $P(-3 < X < 1)$
2. $P(X \ge 1.5)$
3. $P(-5 \le X \le 5)$

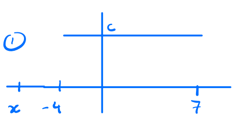
(a)



$$1 = \int_{-\infty}^{\infty} f(t)\,dt = \int_{-4}^{7} c\,dt = 11c$$

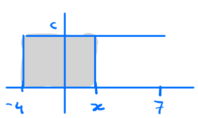
$$\Rightarrow c = \frac{1}{11}$$

(b) (1)



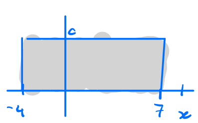
$F(x) = P(X \le x) = 0$

(2)



$F(x) = c(x+4) = \frac{1}{11}(x+4)$

(3)



$F(x) = 1$

$$F(x) = \begin{cases} 0 & x < -4 \\ \frac{1}{11}(x+4) & -4 \le x \le 7 \\ 1 & x > 7 \end{cases}$$

(c) (1) $F(1) - F(-3) = \frac{1}{11}(1+4) - \frac{1}{11}(-3+4) = \frac{5}{11} - \frac{1}{11} = \frac{4}{11}$

(2) $1 - F(1.5) = 1 - \frac{1}{11}(1.5+4) = 1 - \frac{1}{2} = \frac{1}{2}$

(3) $F(5) - F(-5) = \frac{1}{11}(5+4) = \frac{9}{11}$

Problem 2. Consider the continuous random variable X whose density is given by

$$f(x) = \begin{cases} cx^2 & -1 < x \leq 0 \\ x & 0 < x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

a. Draw the graph of f and find c.

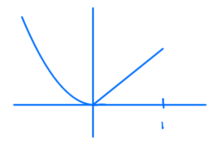
b. Find an expression for $F(x) = P(X \leq x)$ in terms of x when

1. $x \leq -1$
2. $-1 < x \leq 0$
3. $0 < x \leq 1$
4. $x > 1$

c. Use $F(x)$ to find the following probabilities. Do not do any integration.

1. $P(-0.25 \leq X \leq 0.75)$
2. $P(X \leq -0.5)$
3. $P(X > 0.5)$

(a)



$$1 = \int_{-\infty}^{\infty} f(t)\,dt$$

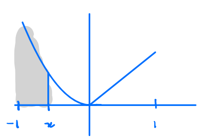
$$= \int_{-1}^{0} ct^2\,dt + \int_{0}^{1} t\,dt$$

$$\frac{1}{2} = \int_{-1}^{0} ct^2\,dt = \frac{c}{3}t^3\Big|_{-1}^{0} = \frac{c}{3}$$

$$\Rightarrow c = \frac{3}{2}$$

(b) (1) $F(x) = 0$

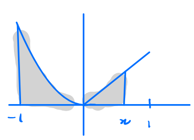
(2)



$$F(x) = \int_{-1}^{x} ct^2\,dt$$

$$= \frac{1}{2}t^3\Big|_{-1}^{x} = \frac{1}{2}(x^3+1)$$

(3)



$$F(x) = \int_{-1}^{0} ct^2\,dt + \int_{0}^{x} t\,dt$$

$$= \frac{1}{2} + \frac{1}{2}t^2\Big|_{0}^{x}$$

$$= \frac{1}{2} + \frac{1}{2}x^2$$

(4) $F(x) = 1$

$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2}(x^3+1) & -1 < x < 0 \\ \frac{1}{2} + \frac{1}{2}x^2 & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

(c) (1) $F(0.75) - F(-0.25) = \frac{1}{2} + \frac{1}{2}(3/4)^2 - \frac{1}{2}((-1/4)^3 + 1)$

$$= \frac{9}{32} + \frac{1}{128} = \frac{37}{128}$$

(2) $F(-0.5) = \frac{1}{2}\left((-\frac{1}{2})^3 + 1\right) = \frac{1}{2}\left(1 - \frac{1}{8}\right) = \frac{7}{16}$

(3) $1 - F(0.5) = 1 - \left(\frac{1}{2} + \frac{1}{2}(\frac{1}{2})^2\right) = 1 - \left(\frac{1}{2} + \frac{1}{8}\right) = \frac{3}{8}$

Problem 3. Suppose X is a continuous random variable with cdf given by

$$F(x) = P(X \leq x) = \begin{cases} 0 & x \leq 0, \\ \frac{x^4}{16} & 0 < x < 2, \\ 1 & x \geq 2. \end{cases}$$

a. Draw a plot of $F(x)$.

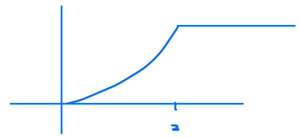
b. Find $f(x)$ and draw a plot on a separate set of axes.

c. How is the area under f related to F?

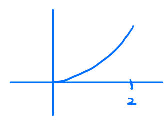
d. Use $F(x)$ to find the following probabilities. Do not do any integration.

1. $P(X > 1)$
2. $P(1 \leq X \leq 2)$
3. $P(1/2 < X < 10)$

(a)



(b) $f(x) = F'(x) = \begin{cases} \frac{1}{4}x^3 & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$



(c) area under f from $-\infty$ to x is $F(x)$

(d) ① $1 - F(1) = 1 - \frac{1}{16} = \frac{15}{16}$

② $F(2) - F(1) = 1 - \frac{1}{16} = \frac{15}{16}$

③ $F(10) - F(\frac{1}{2}) = 1 - \frac{(1/2)^4}{16} = \frac{255}{256}$