§6.4 Expectation and variance

Def If X has density f, then its <u>expected value</u> (or <u>expectation</u> or <u>average value</u> or <u>mean</u>) is given by

$$E[X] = \underbrace{\int_{-\infty}^{\infty}}_{\text{"sum"}} \underbrace{x}_{\text{"possible } X \text{ values"}} \underbrace{f(x)\,dx}_{\text{"weighted by prob."}}$$

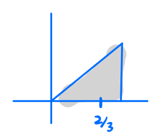
More generally, $E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx$.

For example $E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx$, so we can define

$V(X) = E[X^2] - E[X]^2$ like in the discrete case.

Example Suppose X has density $f(x) = \begin{cases} 2x & 0<x<1 \\ 0 & \text{otherwise.} \end{cases}$

Find $E[3X+4]$ and $V(-2X+5)$.



$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 2x^2\,dx = \frac{2}{3}x^3 \Big|_0^1 = \frac{2}{3}$

So $E[3X+4] = 3E[X]+4 = 3\left(\frac{2}{3}\right)+4 = 6.$

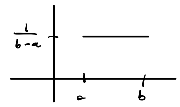
$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 2x^3\,dx = \frac{1}{2}x^4 \Big|_0^1 = \frac{1}{2}$

So $V(X) = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{9}{18} - \frac{8}{18} = \frac{1}{18}$

and $V(-2X+5) = 4V(X) = \frac{4}{18} = \frac{2}{9}$

Def Let $a, b \in \mathbb{R}$ such that $a < b$. A continuous random variable X has the uniform distribution on (a, b) if it has density

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise.} \end{cases}$$



Shorthand: $X \sim \text{Unif}(a, b)$

Example Let $X \sim \text{Unif}(a, b)$. Find $E[X]$

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_a^b x \cdot \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a} \cdot \frac{1}{2} x^2 \Big|_a^b$$

$$= \frac{1}{b-a} \cdot \frac{1}{2}(b^2 - a^2)$$

$$= \frac{a+b}{2}$$

Example Suppose X has density $f(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & x \leq 0. \end{cases}$

Find $E[X]$.

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_0^{\infty} 2x e^{-2x}\,dx$$

int. by parts: $\int u\,dv = uv - \int v\,du$

$u = 2x \qquad dv = e^{-2x}\,dx$

$du = 2\,dx \qquad v = -\frac{1}{2} e^{-2x}$

$$= \lim_{b \to \infty} \int_0^b 2x e^{-2x}\,dx$$

$$= \lim_{b \to \infty} \left(-x e^{-2x} \Big|_0^b + \int_0^b e^{-2x}\,dx \right)$$

$$= \lim_{b \to \infty} \left(-b e^{-2b} - \frac{1}{2} e^{-2x} \Big|_0^b \right)$$

$$= \lim_{b \to \infty} \left(-b e^{-2b} - \frac{1}{2} e^{-2b} + \frac{1}{2} \right)$$

$$= \frac{1}{2}$$

Problem 1. Suppose X is a continuous random variable with density

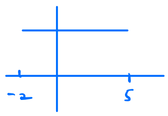
$$f(x) = \begin{cases} c & -2 \leq x \leq 5 \\ 0 & \text{otherwise.} \end{cases}$$

a. Draw a graph of f and find c.

b. Give the name of the distribution of X.

c. Find $E[X]$ and $V(X)$.

(a)



$$1 = \int_{-2}^{5} c\,dx = 7c \implies c = \frac{1}{7}$$

(b) Unif $[-2,5]$

(c) $E[X] = \frac{-2+5}{2} = 1.5$ $\quad E[X^2] = \int_{-2}^{5} x^2 \cdot \frac{1}{7}\,dx = \frac{1}{21} x^3 \Big|_{-2}^{5} = \frac{1}{21}(125+8)$

$$= \frac{133}{21}$$

So $V(X) = \frac{133}{21} - (1.5)^2 = \frac{49}{12}$

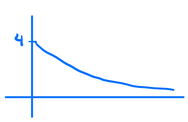
Problem 2. Suppose X is a continuous random variable with density

$$f(x) = \begin{cases} 4e^{-4x} & x > 0 \\ 0 & x \leq 0. \end{cases}$$

a. Draw a graph of f.

b. Find $E[X]$ and $V(X)$.

(a)



(b) $E[X] = \int_0^\infty 4xe^{-4x}\,dx$ $\quad u = 4x \quad dv = e^{-4x}dx$

$du = 4dx \quad v = -\frac{1}{4}e^{-4x}$

$$= \lim_{b\to\infty} \left(-xe^{-4x} \Big|_0^b + \int_0^b e^{-4x}\,dx \right)$$

$$= \lim_{b\to\infty} \left(-be^{-4b} - \frac{1}{4}e^{-4x} \Big|_0^b \right)$$

$$= \lim_{b\to\infty} \left(-be^{-4b} - \frac{1}{4}e^{-4b} + \frac{1}{4} \right)$$

$$= \frac{1}{4}$$

$E[X^2] = \int_0^\infty 4x^2 e^{-4x}\,dx$ $\quad u = 4x^2 \quad dv = e^{-4x}dx$

$du = 8x\,dx \quad v = -\frac{1}{4}e^{-4x}$

$$= \lim_{b\to\infty} \left(-x^2 e^{-4x} \Big|_0^b + \int_0^b 2xe^{-4x}\,dx \right)$$

$u = 2x \quad dv = e^{-4x}dx$

$du = 2dx \quad v = -\frac{1}{4}e^{-4x}$

$$= \lim_{b\to\infty} \left(-b^2 e^{-4b} - \frac{1}{2}xe^{-4x} \Big|_0^b + \int_0^b \frac{1}{2}e^{-4x}\,dx \right)$$

$$= \lim_{b \to \infty} \left(-b^2 e^{-4b} - \frac{1}{2} b e^{-4b} - \frac{1}{8} e^{-4x} \Big|_0^b \right)$$

$$= \frac{1}{8}$$

$$V(X) = \frac{1}{8} - \left(\frac{1}{4}\right)^2 = \frac{2}{16} - \frac{1}{16} = \frac{1}{16}$$

Problem 3. Let X denote the weekly CPU time used by an accounting firm (measured in hours), and suppose X has pdf given by

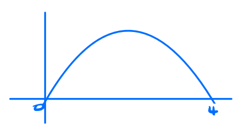
$$f(x) = \begin{cases} \frac{3}{32} x(4-x) & 0 \le x \le 4 \\ 0 & \text{otherwise.} \end{cases}$$

a. Draw a graph of f.

b. Find $E[X]$ and $V(X)$.

c. The CPU time costs the firm 200 dollars an hour, plus a fixed base cost of 50 dollars per week. Let C denote the total cost in a given week. Find $E[C]$ and $V(C)$.

a



b

$$E[X] = \int_0^4 \frac{3}{32} x^2 (4-x)\, dx$$

$$= \frac{3}{32} \int_0^4 (4x^2 - x^3)\, dx$$

$$= \frac{3}{32} \left(\frac{4}{3} x^3 - \frac{1}{4} x^4 \Big|_0^4 \right)$$

$$= \frac{3}{32} \left(\frac{4^4}{3} - \frac{4^4}{4} \right)$$

$$= 2$$

$$E[X^2] = \int_0^4 \frac{3}{32} x^3 (4-x)\, dx$$

$$= \frac{3}{32} \int_0^4 (4x^3 - x^4)\, dx$$

$$= \frac{3}{32} \left(x^4 - \frac{1}{5} x^5 \Big|_0^4 \right)$$

$$= \frac{3}{32} \left(4^4 - \frac{4^5}{5} \right)$$

$$= \frac{24}{5}$$

$$V(X) = \frac{24}{5} - 2^2 = \frac{4}{5}$$

c $C = 200X + 50$, $E[C] = 200E[X] + 50 = 200(2) + 50 = 450$

$$V(C) = 200^2 V(X) = 200^2 \left(\frac{4}{5}\right) = 32000$$