§ 6.5 Exponential Distribution

Def A random variable X has the exponential distribution with parameter $\lambda > 0$ if it has density $f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 . \end{cases}$

Shorthand: $X \sim \mathrm{Exp}(\lambda)$

Mean: $E[X] = \frac{1}{\lambda}$, Variance: $V(X) = \frac{1}{\lambda^2}$.

Derivation: Let $Y \sim \mathrm{Pois}(\lambda)$ be the number of arrivals (eg. phone calls) received in one time unit (eg. 1 hour), where λ is the average number of arrivals in one time unit. Similarly, $Y_t \sim \mathrm{Pois}(\lambda t)$ models number of arrivals over t time units, where $t > 0$.

Let X be the time until the first arrival

Then X is a continuous random variable and for any $t > 0$,

$$\begin{aligned} &P(X > t) \\ &= P(\text{no arrivals in } t \text{ time units}) \\ &= P(Y_t = 0) \\ &= e^{-\lambda t}. \end{aligned}$$

Therefore, X has CDF given by

$$F(t) = P(X \leq t) = 1 - P(X > t) = 1 - e^{-\lambda t}$$

when $t > 0$ (and $F(t) = 0$ when $t \leq 0$), and density

$$f(t) = F'(t) = \begin{cases} \lambda e^{-\lambda t} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

Remarks

1. $X \sim Exp(\lambda)$ models the waiting time until an arrival.

2. If λ is the average number of arrivals per hour, then $1/\lambda$ is the average number of hours per arrival.

3. The set $\{Y_t : t > 0\}$, which models arrivals over arbitrary time intervals is called a Poisson process.

4. The exponential distribution can be thought of as the continuous analogue of the geometric distribution.

Example Suppose customers arrive at a salon according to a Poisson process with an average of 3 customers per hour. Let X be the time until the next customer arrives. Find

$3 = \lambda$

1. the probability it takes between 30 and 90 minutes for the next customer to arrive

customers per hour

$X \sim Exp(3)$

units of X are hours

$$P(\tfrac{1}{2} \le X \le \tfrac{3}{2}) = F(3/2) - F(1/2)$$

hours

$$= (1 - e^{-3/2\lambda}) - (1 - e^{-1/2\lambda})$$

$$= e^{-1/2\lambda} - e^{-3/2\lambda}$$

$$= e^{-3/2} - e^{-9/2}$$

② the probability it takes more than a 1/2 hour until the next customer arrives

$$P(X > 1/2) = 1 - F(1/2) = e^{-3/2}$$

③ the probability it takes more than 1 hour given no one has arrived in 1/2 hour

$$P(X > 1 \mid X > 1/2) = \frac{P(X > 1, X > 1/2)}{P(X > 1/2)}$$

$$= \frac{P(X > 1)}{P(X > 1/2)}$$

$$= \frac{e^{-\lambda}}{e^{-1/2\lambda}}$$

$$= e^{-1/2\lambda} = e^{-3/2}$$

④ mean time until next customer arrives.

$$E[X] = \frac{1}{\lambda} = \frac{1}{3} \text{ hour}$$

Theorem (memoryless property) Let $X \sim Exp(\lambda)$.

Then $$\underbrace{P(X > t+s \mid X > s)}_{\text{prob. of waiting } t \text{ more time units given that you've already waited } s \text{ time units}} = \underbrace{P(X > t)}_{\text{prob. of waiting } t \text{ time units.}}.$$

Problem 1. Let $X \sim \text{Exp}(5)$ be the time in hours until you receive a spam email message.

a. Find the probability it takes less than 2 hours to receive a spam message.

b. Find the probability it takes between .75 and 1.3 hours to receive a spam message.

c. Find the expected time in hours until you receive a spam message.

d. Find the expected time in minutes until you receive a spam message.

e. Find the average rate, in messages per hour, at which you receive spam messages.

f. Find the average rate, in messages per minute, at which you receive spam messages.

$\lambda = 5$

a $P(X \le 2) = F(2) = 1 - e^{-2\lambda} = 1 - e^{-10}$

b $P(0.75 \le X \le 1.3) = F(1.3) - F(0.75) = (1 - e^{-1.3\lambda}) - (1 - e^{-0.75\lambda})$

$= e^{-3.75} - e^{-6.5}$

c $E[X] = 1/\lambda = 1/5$ hour

d $E[60X] = 60(1/5) = 12$ minutes

e 5 messages per hour

f $5/60 = 1/12$ messages per minute

Problem 2. Suppose that phone calls arrive at a call center, which opens at 8:00 a.m., according to a Poisson process with an average rate of 1 call every 15 minutes. Let X denote the amount of time elapsed since 8:00 a.m. until the first phone call.

a. What is the value of λ if using units of hours? Minutes?

b. Use units of minutes to find the probability that the first phone call occurs

1. after 8:30 a.m.
2. before 8:20 a.m.
3. between 8:15 and 8:45 a.m.

c. How do your answers change if you convert your work to hours?

d. Suppose it's now 9:00 a.m. and no phone calls have come. Given this, what is the probability that you receive no calls in the next 15 minutes?

e. Suppose you want to sleep in a little and arrive at the call center after 8:00 a.m., knowing that it's unlikely for the first call of the day to come in right when the center opens. You don't want to miss the first call however. How late can you be so that you're 90% sure you get in before the first call?

f. Find $E[X]$ and find the expected time of the first call.

g. Suppose I tell you that on average there's 45 minutes between phone calls. What is the value of λ in this case? What is $E[X]$?

ⓐ $\lambda = 4$ calls per hour or $\lambda = 4/60 = 1/15$ calls per minute

ⓑ ① $P(X > 30) = 1 - P(X \le 30)$

$$= 1 - F(30) = e^{-30\lambda} = e^{-30(1/15)} = e^{-2}$$

② $P(X < 20) = F(20) = 1 - e^{-20\lambda} = 1 - e^{-20(1/15)} = 1 - e^{-4/3}$

③ $P(15 < X < 45) = F(45) - F(15) = e^{-15\lambda} - e^{-45\lambda}$

$$= e^{-1} - e^{-3}$$

ⓒ They don't change:

① $P(X > 1/2) = 1 - F(1/2) = 1 - e^{-1/2\,\lambda} = 1 - e^{-1/2(4)} = 1 - e^{-2}$

② $P(X < 1/3) = F(1/3) = 1 - e^{-1/3\lambda} = 1 - e^{-4/3}$

③ $P(1/4 < X < 3/4) = F(3/4) - F(1/4) = e^{-1/4\lambda} - e^{-3/4\lambda}$

$$= e^{-1} - e^{-3}$$

ⓓ memoryless property $\Rightarrow$ $P(X > 75 \mid X > 60) = P(X > 15)$

$$= 1 - F(15)$$

$$= e^{-15\lambda} = e^{-1}$$

ⓔ find t so that $0.9 = P(X > t)$

$$= 1 - F(t) = e^{-\lambda t}$$

$$\Rightarrow \ln(0.9) = -\lambda t$$

$$\Rightarrow t = -\frac{1}{\lambda}\ln(0.9) = -15\ln(0.9) \text{ minutes}$$

(or $-\frac{1}{4}\ln(0.9)$ hours)

ⓕ $E[X] = 1/\lambda = 15$ minutes, 8:15 a.m.

ⓖ $\lambda = 1/45$ calls per minute, $E[X] = 45$ minutes.