§6.6 Joint distributions

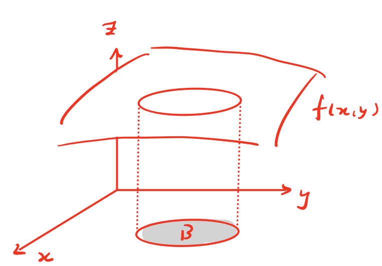
Def The joint probability density function of a pair of continuous random variables X and Y is a function $f: \mathbb{R}^2 \to \mathbb{R}$ with the properties:

① $f(x,y) \geq 0$ for all $x, y \in \mathbb{R}$

② $\iint_{\mathbb{R}^2} f(x,y)\, dA = 1$ (ie $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dy\, dx = 1$)

③ for any set $B \subseteq \mathbb{R}^2$, $P((X,Y) \in B) = \iint_B f(x,y)\, \underbrace{dA}_{\substack{= dx\,dy \\ \text{or } dy\,dx}}$.

Remarks The graph of $z = f(x,y)$ is a surface in $\mathbb{R}^3$, $B \subseteq \mathbb{R}^2$ is a region in the xy-plane, and $P((X,Y) \in B)$ is the volume below $z = f(x,y)$ and above B.

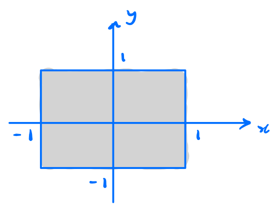


Multivariable integration review

something non-zero

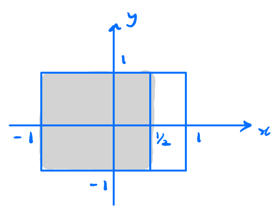
Example Let $f(x,y) = \begin{cases} \boxed{} & -1<x<1,\ -1<y<1 \\ 0 & \text{otherwise.} \end{cases}$

① Sketch the region in $\mathbb{R}^2$ (the xy-plane) where $f(x,y) \neq 0$. This is called the support of f and is denoted supp(f).



② Let $A = \{(x,y) \in \mathbb{R}^2 : x < \frac{1}{2}\}$. Sketch the region $A \cap \text{supp}(f)$ and set up an integral for $\iint_A f(x,y)\, dA$ in two ways: ① $dA = dx\,dy$

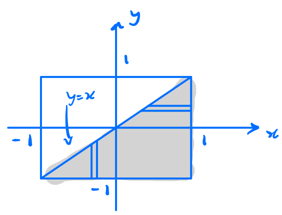
② $dA = dy\,dx$



$$\int_{x=-1}^{x=1/2} \int_{y=-1}^{y=1} \boxed{}\, dy\,dx = \int_{y=-1}^{y=1} \int_{x=-1}^{x=1/2} \boxed{}\, dx\,dy$$

Remark This integral represents $P(X < 1/2)$

③ Let $B = \{ (x,y) \in \mathbb{R}^2 : x > y \}$. Sketch $B \cap \text{supp}(f)$ and set up $\iint_B f(x,y)\, dA$ in two ways again.



$$\int_{x=-1}^{x=1} \int_{y=1}^{y=x} f(x,y)\, dy\, dx$$

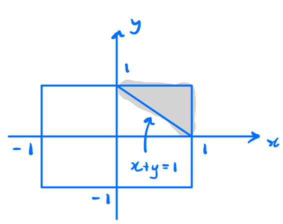
$$= \int_{y=-1}^{y=1} \int_{x=y}^{x=1} f(x,y)\, dx\, dy$$

represents $P(X > Y)$

Recall ① outside limits of integration must be constants

② inside limits of integration can depend on the outside limit variable.

④ Let $C = \{ (x,y) \in \mathbb{R}^2 : x+y > 1 \}$. Sketch $C \cap \text{supp}(f)$ and set up $\iint_C f(x,y)\, dA$ in two ways.

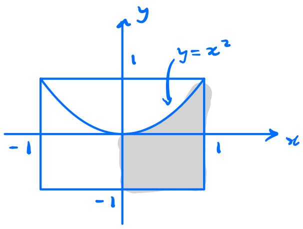


$$\int_{x=0}^{x=1} \int_{y=1-x}^{y=1} f(x,y)\, dy\, dx$$

$$= \int_{y=0}^{y=1} \int_{x=1-y}^{x=1} f(x,y)\, dx\, dy$$

represents $P(X + Y > 1)$

⑤ Let $D = \{ (x,y) \in \mathbb{R}^2 : x > 0,\ y < x^2 \}$. Sketch $D \cap \mathrm{supp}(f)$ and set up $\iint_D f(x,y)\,dA$ in two ways.



$$\int_{x=0}^{x=1} \int_{y=-1}^{y=x^2} f(x,y)\,dy\,dx$$

$$= \int_{y=-1}^{y=0} \int_{x=0}^{x=1} f(x,y)\,dx\,dy + \int_{y=0}^{y=1} \int_{x=\sqrt{y}}^{x=1} f(x,y)\,dx\,dy$$

represents $P(X > 0,\ Y < X^2)$

⑥ Suppose $\boxed{} = cx^2y^4$ and $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$.

Find c.

$$1 = \int_{x=-1}^{x=1} \int_{y=-1}^{y=1} cx^2y^4\,dy\,dx$$

$$= c\int_{-1}^{1} x^2 \left(\frac{1}{5} y^5 \Big|_{-1}^{1} \right) dx$$

$$= \frac{2}{5} c \int_{-1}^{1} x^2\,dx$$

$$= \frac{2}{5} c \left(\frac{1}{3} x^3 \Big|_{-1}^{1} \right)$$

$$= \frac{4}{15} c, \quad \text{So} \quad c = \frac{15}{4}.$$

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a given function where

$$\operatorname{supp}(f) = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < 1\},$$

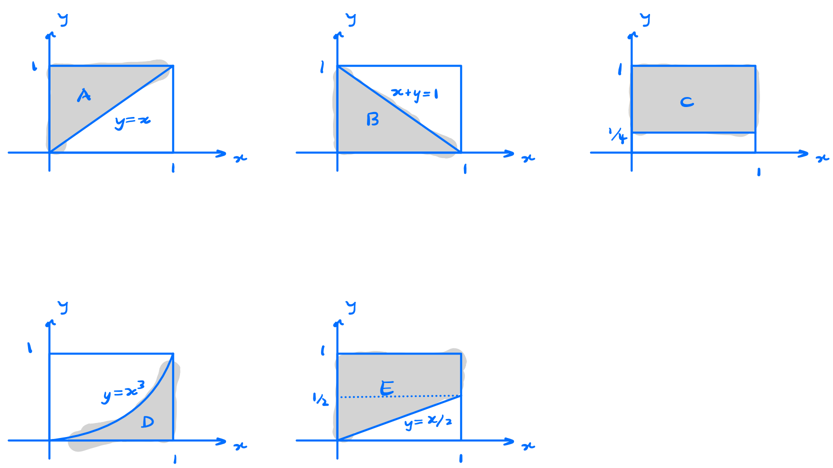
and consider the following given sets:

$$\begin{aligned}
A &= \{(x, y) \in \mathbb{R}^2 : x < y\}\\
B &= \{(x, y) \in \mathbb{R}^2 : x + y < 1\}\\
C &= \{(x, y) \in \mathbb{R}^2 : y > 1/4\}\\
D &= \{(x, y) \in \mathbb{R}^2 : y < x^3\}\\
E &= \{(x, y) \in \mathbb{R}^2 : y > x/2\}.
\end{aligned}$$

Problem 1. For each set:

a. Sketch its intersection with $\operatorname{supp}(f)$ on the xy-plane.

b. Set up a double integral that gives the volume under f and above the set in two ways: with $dA = dydx$ and with $dA = dxdy$.



$$\iint_A f(x,y)\, dA = \int_{x=0}^{x=1} \int_{y=x}^{y=1} f(x,y)\, dy dx = \int_{y=0}^{y=1} \int_{x=0}^{x=y} f(x,y)\, dx dy$$

$$\iint_B f(x,y)\, dA = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} f(x,y)\, dy dx = \int_{y=0}^{y=1} \int_{x=0}^{x=1-y} f(x,y)\, dx dy$$

$$\iint_C f(x,y)\, dA = \int_{x=0}^{x=1} \int_{y=1/4}^{y=1} f(x,y)\, dy dx = \int_{y=1/4}^{y=1} \int_{x=0}^{x=1} f(x,y)\, dx dy$$

$$\iint_D f(x,y)\, dA = \int_{x=0}^{x=1} \int_{y=0}^{y=x^3} f(x,y)\, dy dx = \int_{y=0}^{y=1} \int_{x=y^{1/3}}^{x=1} f(x,y)\, dx dy$$

$$\iint_E f(x,y)\,dA = \int_{x=0}^{x=1}\int_{y=x/2}^{y=1} f(x,y)\,dy\,dx = \int_{y=0}^{y=1/2}\int_{x=0}^{x=2y} f(x,y)\,dx\,dy + \int_{y=1/2}^{y=1}\int_{x=0}^{x=1} f(x,y)\,dx\,dy$$

Problem 2. Suppose $f(x,y) = cxy$ on its support.

a. Find the value of c so that the total volume under f is 1.

b. Compute $\iint_E f(x,y)\,dA$ where E is the set given above.

ⓐ
$$1 = \int_{x=0}^{x=1}\int_{y=0}^{y=1} cxy\,dy\,dx = c\int_{x=0}^{1} x\left(\tfrac{1}{2}y^2\Big|_0^1\right)dx$$
$$= \tfrac{1}{2}c\left(\tfrac{1}{2}x^2\Big|_0^1\right)$$
$$= \tfrac{1}{4}c \implies c = 4$$

ⓑ
$$\int_0^1\int_{x/2}^1 4xy\,dy\,dx = 4\int_0^1 x\left(\tfrac{1}{2}y^2\Big|_{x/2}^1\right)dx$$
$$= 2\int_0^1 x\left(1 - \tfrac{x^2}{4}\right)dx$$
$$= \int_0^1 \left(2x - \tfrac{1}{2}x^3\right)dx$$
$$= x^2 - \tfrac{1}{8}x^4\Big|_0^1$$
$$= \tfrac{7}{8}$$