# §6.6 Joint densities

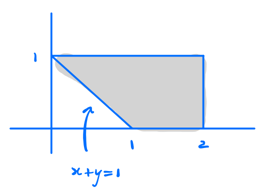
Recall that if $f(x,y)$ is the joint density of $X$ and $Y$,

$$P((X,Y) \in B) = \iint_B f(x,y)\, dA.$$

Example Let $f(x,y) = \begin{cases} cxy & 0<x<2,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$

for some constant $c$. Express the following probabilities as double integrals:

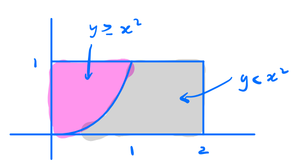
① $P(X+Y>1)$




$$\int_{y=0}^{y=1} \int_{x=1-y}^{x=2} cxy\, dx\, dy$$

$$= \int_{x=0}^{x=1} \int_{y=1-x}^{y=1} cxy\, dy\, dx + \int_{x=1}^{x=2} \int_{y=0}^{y=1} cxy\, dy\, dx$$

② $P(Y < X^2)$ $= 1 - P(Y \geq X^2)$




$$= 1 - \int_{x=0}^{x=1} \int_{y=x^2}^{y=1} cxy\, dy\, dx$$

$$= 1 - \int_{y=0}^{y=1} \int_{x=0}^{x=\sqrt{y}} cxy\, dx\, dy$$

③ Find $c$.

$$1 = \int_{x=0}^{x=2} \int_{y=0}^{y=1} cxy\, dy\, dx = c \int_{x=0}^{x=2} x \left( \int_{y=0}^{y=1} y\, dy \right) dx$$

$$= c \int_0^2 x \left( \tfrac{1}{2} y^2 \Big|_0^1 \right) dx$$

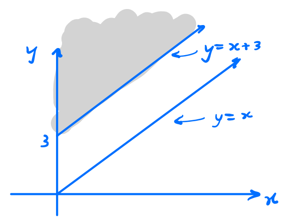
$$= \tfrac{1}{2} c \int_0^2 x\, dx$$

$$= \tfrac{1}{4} c\, x^2 \Big|_0^2 = c, \quad \text{So } c = 1.$$

Example Suppose an industrial manufacturing system has two components A and B whose lifetimes are $X$ and $Y$ respectively with joint density $f(x,y) = \begin{cases} e^{-y} & 0<x<y<\infty \\ 0 & \text{otherwise.} \end{cases}$

① Find the probability that component B lasts at least 3 time units longer than component A.

$$P(Y \geq X+3) = \int_{x=0}^{\infty} \int_{y=x+3}^{\infty} e^{-y}\, dy\, dx$$

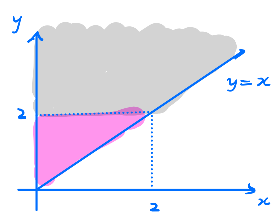



$$= -\int_0^{\infty} \left( e^{-y} \Big|_{x+3}^{\infty} \right) dx$$

$$= \int_0^{\infty} e^{-(x+3)}\, dx = -e^{-(x+3)} \Big|_0^{\infty} = e^{-3}$$

② Find the probability that at least 1 component lasts at least 2 time units.

$$P(X \geq 2 \text{ or } Y \geq 2)$$




$$= 1 - P(X < 2, Y < 2)$$

$$= 1 - \int_{x=0}^{x=2} \int_{y=x}^{y=2} e^{-y}\, dy\, dx$$

$$= 1 + \int_0^2 \left( e^{-y} \Big|_x^2 \right) dx$$

$$= 1 + \int_0^2 (e^{-2} - e^{-x})\, dx$$

$$= 1 + 2e^{-2} - \int_0^2 e^{-x}\, dx$$

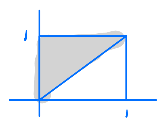
$$= 1 + 2e^{-2} + e^{-x} \Big|_0^2$$

$$= 1 + 2e^{-2} + e^{-2} - 1 = 3e^{-2}$$

**Expectations** $E[g(X,Y)] = \iint_{\mathbb{R}^2} g(x,y) f(x,y)\, dA$

**Example** Compute $E[XY]$ using the joint density

$$f(x,y) = \begin{cases} 21x^2y^3 & 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$



$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$$

$$= \int_{y=0}^{y=1} \int_{x=0}^{x=y} 21\, x^3 y^4 \, dx\, dy$$

$$= 21 \int_0^1 y^4 \left( \frac{1}{4} x^4 \Big|_0^y \right) dy$$

$$= \frac{21}{4} \int_0^1 y^8\, dy$$

$$= \frac{21}{36} = \frac{7}{12}$$

**Problem 1.** Let $X$ and $Y$ have joint density given by

$$f(x,y) = \begin{cases} 6(x-y) & 0 \le y \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

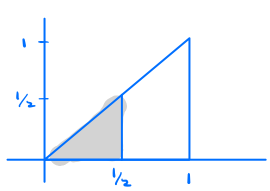
Express each of the following probabilities as a double integral. Make sure you can compute these by hand, but save that for later. You may use Wolfram Alpha to check your answer when you do so.

a. $P(X \le 1/2)$

b. $P(Y < X^3)$

c. $P(X + Y < 1)$

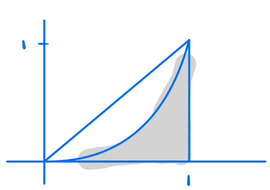
(a)



$$\int_{x=0}^{x=1/2} \int_{y=0}^{y=x} 6(x-y)\, dy\, dx$$

$$= \int_{y=0}^{y=1/2} \int_{x=y}^{x=1/2} 6(x-y)\, dx\, dy$$

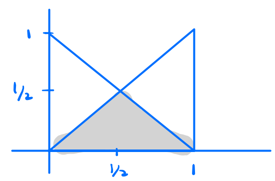
(b)



$$\int_{x=0}^{x=1} \int_{y=0}^{y=x^3} 6(x-y)\, dy\, dx$$

$$= \int_{y=0}^{y=1} \int_{x=y^{1/3}}^{x=1} 6(x-y)\, dx\, dy$$

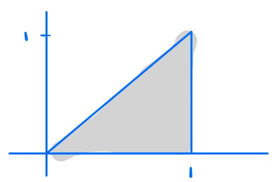
(c)




$$\int_{x=0}^{x=1/2}\int_{y=0}^{y=x} 6(x-y)\,dy\,dx + \int_{x=1/2}^{x=1}\int_{y=0}^{y=1-x} 6(x-y)\,dy\,dx$$

$$= \int_{y=0}^{y=1/2}\int_{x=y}^{x=1-y} 6(x-y)\,dx\,dy$$

**Problem 2.** Compute $E[X^2Y]$ using the joint density from Problem 1.




$$\int_{x=0}^{x=1}\int_{y=0}^{y=x} 6x^2y\,(x-y)\,dy\,dx$$

$$= 6\int_0^1 x^2 \int_0^x (yx-y^2)\,dy\,dx$$

$$= 6\int_0^1 x^2\left(\tfrac{1}{2}xy^2 - \tfrac{1}{3}y^3\Big|_0^x\right)dx$$

$$= 6\int_0^1 \left(\tfrac{1}{2}x^5 - \tfrac{1}{3}x^5\right)dx$$

$$= \int_0^1 x^5\,dx = \frac{1}{6}$$

**Problem 3.** Let $X$ and $Y$ have joint density given by

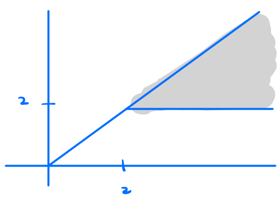
$$f(x,y) = \begin{cases} 2e^{-(x+y)} & 0 < y < x < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Express each of the following probabilities as a double integral and then compute its value using Wolfram Alpha.

a. $P(Y > 2)$

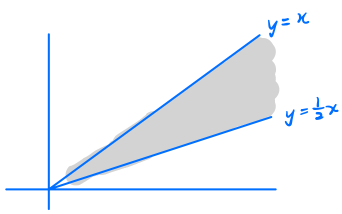
b. $P(2Y > X)$

c. $P(X + Y > 2)$



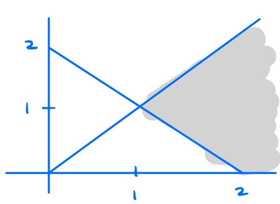

$$\int_{x=2}^{\infty} \int_{y=2}^{y=x} 2e^{-(x+y)}\, dy\, dx$$

$$= \int_{y=2}^{y=\infty} \int_{x=y}^{x=\infty} 2e^{-(x+y)}\, dx\, dy$$




$$\int_{x=0}^{x=\infty} \int_{y=\frac{1}{2}x}^{y=x} 2e^{-(x+y)}\, dy\, dx$$

$$= \int_{y=0}^{y=\infty} \int_{x=y}^{x=2y} 2e^{-(x+y)}\, dx\, dy$$




$$\int_{x=1}^{x=2} \int_{y=2-x}^{y=x} 2e^{-(x+y)}\, dy\, dx$$

$$+ \int_{x=2}^{x=\infty} \int_{y=0}^{y=x} 2e^{-(x+y)}\, dy\, dx$$

$$= \int_{y=0}^{y=1} \int_{x=2-y}^{x=\infty} 2e^{-(x+y)}\, dx\, dy$$

$$+ \int_{y=1}^{y=\infty} \int_{x=y}^{x=\infty} 2e^{-(x+y)}\, dx\, dy$$