§ 6.6, 6.7 Marginal densities and independence

Def Given the joint density $f(x,y)$ of X and Y the marginal density of X is given by

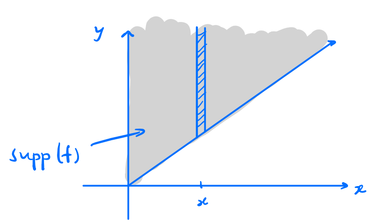
$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

and the marginal density of Y is given by

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx.$$

Remark This is just like summing over rows / columns in joint pmf table for discrete random variables.

Example Let $f(x,y) = \begin{cases} e^{-y} & 0 < x < y < \infty \\ 0 & \text{otherwise} \end{cases}$ be the joint density of X and Y. Find the marginal densities of X and Y.

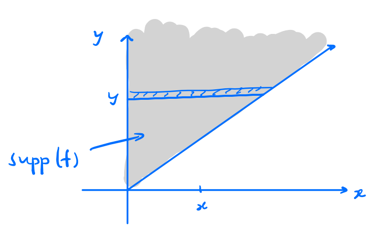


$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{y=x}^{y=\infty} e^{-y}\,dy$$

$$= -e^{-y}\Big|_x^{\infty} = e^{-x} \quad \text{when } x > 0$$

$f_X(x) = 0$ when $x \le 0$.



$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_{x=0}^{x=y} e^{-y}\,dx$$

$$= ye^{-y} \quad \text{when } y > 0$$

$f_Y(y) = 0$ when $y \le 0$.

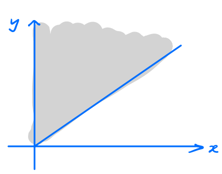
Def Random variables X and Y are <u>independent</u> if their joint density is the product of their marginal densities: $f(x,y) = f_X(x)\, f_Y(y)$ for all $(x,y) \in \mathbb{R}^2$.

Remark The random variables in the previous example are <u>not</u> independent.

Example Caleb is in the regular ticket line at the airport and his waiting time C is exponentially distributed with mean 10 minutes. Destiny is in the express line with waiting time D exp. dist'd with mean 5 minutes. Their waiting times are independent. Find $P(C<D)$.

Since $C \sim \text{Exp}(1/10)$, $D \sim \text{Exp}(1/5)$, are independent, their joint density is

$$f(x,y) = f_C(x)\, f_D(y) = \begin{cases} \frac{1}{50} e^{-\frac{1}{10}x} e^{-\frac{1}{5}y} & x>0,\ y>0 \\ 0 & \text{otherwise} \end{cases}$$



$$P(C<D) = \int_{y=0}^{\infty} \int_{x=0}^{x=y} \frac{1}{50} e^{-\frac{1}{10}x} e^{-\frac{1}{5}y}\, dx\, dy$$

$$= \frac{1}{50} \int_0^\infty e^{-\frac{1}{5}y} \left(-10 e^{-\frac{1}{10}x} \Big|_0^y \right) dy$$

$$= \frac{1}{5} \int_0^\infty e^{-\frac{1}{5}y} \left(1 - e^{-\frac{1}{10}y} \right) dy$$

$$= \frac{1}{5} \int_0^\infty \left(e^{-\frac{1}{5}y} - e^{-\frac{3}{10}y} \right) dy$$

$$= \frac{1}{5} \left(5 - \frac{10}{3} \right) = \frac{1}{3}$$

Problem 1. Let X and Y have joint density given by

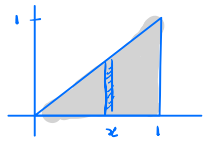
$$f(x,y) = \begin{cases} 6(x-y) & 0 \le y \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

a. Find the marginal density of X. Make sure to give a piecewise definition.

b. Find the marginal density of Y. Make sure to give a piecewise definition.

c. Are X and Y independent?

(a)



when $0 < x < 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_0^x 6(x-y)\,dy$$

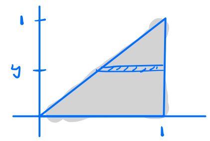
$$= 6\left(xy - \tfrac{1}{2}y^2 \Big|_0^x\right)$$

$$= 6\left(x^2 - \tfrac{1}{2}x^2\right)$$

$$= 3x^2$$

So $f_X(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

(b)



when $0 < y < 1$,

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_y^1 6(x-y)\,dx$$

$$= 6\left(\tfrac{1}{2}x^2 - xy \Big|_y^1\right)$$

$$= 6\left(\tfrac{1}{2} - y - \tfrac{1}{2}y^2 + y^2\right)$$

$$= 3 - 6y + 3y^2$$

$$= 3(y-1)^2$$

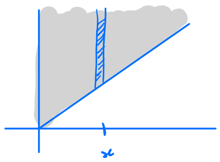
So $f_Y(y) = \begin{cases} 3(y-1)^2 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$

(c) No, since $f(x,y) \neq f_X(x) f_Y(y)$

Problem 2. Answer the questions of Problem 1 using the joint density

$$f(x,y) = \begin{cases} 2e^{-(x+y)} & 0 < x < y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

(a)



$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_x^{\infty} 2e^{-(x+y)}\,dy$$

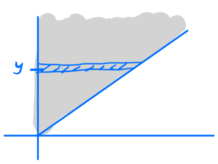
$$= 2e^{-x}\int_x^{\infty} e^{-y}\,dy$$

$$= 2e^{-x}\left(-e^{-y}\Big|_x^{\infty}\right)$$

$$= 2e^{-x}(e^{-x}) = 2e^{-2x}$$

So $f_X(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & x \le 0 \end{cases}$ (note $X \sim \text{Exp}(2)$)

(b)



when $y > 0$,

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^{y} 2e^{-(x+y)}\,dx$$

$$= 2e^{-y}\int_0^{y} e^{-x}\,dx$$

$$= 2e^{-y}\left(-e^{-x}\Big|_0^{y}\right)$$

$$= 2e^{-y}(1-e^{-y})$$

So $f_Y(y) = \begin{cases} 2e^{-y}(1-e^{-y}) & y > 0 \\ 0 & y \le 0 \end{cases}$

(c) No, since $f(x,y) \neq f_X(x)\, f_Y(y)$

Problem 3. Suppose Xavier and Yolanda are planning to meet at a coffee shop for lunch. Xavier will arrive at some time uniformly distributed between 1:00 and 1:30 p.m, and Yolanda will arive at some time uniformly distributed between 1:00 and 1:20. Their arrival times are independent.

a. What is the probability that Xavier arrives at least 10 minutes after Yolanda?

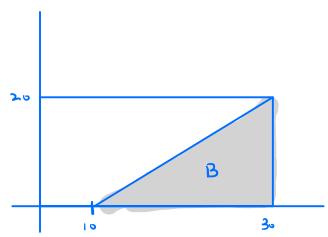
b. What is the probability that they arrive within 5 minutes of each other?

$$f_X(x) = \begin{cases} \frac{1}{30} & 0 < x < 30 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{20} & 0 < y < 20 \\ 0 & \text{otherwise} \end{cases}$$

$$f(x,y) = f_X(x)\, f_Y(y)$$

$$= \begin{cases} \frac{1}{600} & 0 < x < 30,\ 0 < y < 20 \\ 0 & \text{otherwise} \end{cases}$$

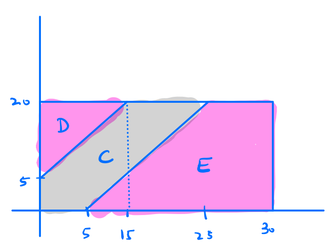


$$P(X > Y + 10)$$

$$= \iint_B f(x,y)\, dA$$

$$= \frac{1}{600} \iint_B dA = \frac{\text{area}(B)}{600}$$

$$= \frac{\frac{1}{2}(20)^2}{600} = \frac{1}{3}$$



$x - y = 5$

$x - y = -5$

$$P(|X - Y| < 5)$$

$$= P(-5 < X - Y < 5)$$

$$= 1 - \frac{\text{area}(D)}{600} - \frac{\text{area}(E)}{600}$$

$$= 1 - \frac{\frac{1}{2}(15)^2}{600} - \frac{\frac{1}{2}(20)^2 + 100}{600}$$

$$= 1 - \frac{225}{1200} - \frac{600}{1200}$$

$$= \frac{375}{1200} = 0.3125$$