§7.1 MGF's and Sums of Normal Random Variables

Theorem Let $Z \sim N(0,1)$. Then the mgf of Z is $m_Z(t) = e^{t^2/2}$.

Proof Observe that

$$m_Z(t) = E\left[e^{tZ}\right]$$

$$= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2 + tx} \, dx$$

Note that $-\frac{1}{2}x^2 + tx = -\frac{1}{2}(x^2 + 2tx)$

$$= -\frac{1}{2}(x^2 + 2tx + t^2) + \frac{1}{2}t^2$$

$$= -\frac{1}{2}(x+t)^2 + \frac{1}{2}t^2$$

Therefore $e^{-\frac{1}{2}x^2 + tx} = e^{-\frac{1}{2}(x+t)^2} \cdot e^{\frac{1}{2}t^2}$

and $m_Z(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x+t)^2} \cdot e^{\frac{1}{2}t^2} \, dx$

$$= e^{\frac{1}{2}t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x+t)^2} \, dx$$

$$= e^{\frac{1}{2}t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \, dx$$

$$= e^{\frac{1}{2}t^2}$$

Problem 1. Let $X \sim N(\mu, \sigma^2)$. Use the moment generating function of $Z \sim N(0,1)$ to find the moment generating function of X. *Hint*: remember that $Z = (X - \mu)/\sigma$.

$X = \mu + \sigma Z$ so

$$\begin{aligned} m_X(t) &= E[e^{tX}] \\ &= E[e^{t(\mu+\sigma Z)}] \\ &= E[e^{\mu t} e^{(t\sigma)Z}] \\ &= e^{\mu t} m_Z(t\sigma) \\ &= e^{\mu t} e^{\frac{1}{2}\sigma^2 t^2} \end{aligned}$$

Problem 2. Let $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ be independent normal random variables.

a. Find the moment generating function of $X + Y$.

b. Explain why $X + Y$ is normally distributed and give its mean and variance.

$$\begin{aligned} m_{X+Y}(t) &= E[e^{t(X+Y)}] \\ &= E[e^{tX}]\,E[e^{tY}] \\ &= e^{\mu_X t} e^{\frac{1}{2}\sigma_X^2 t^2} e^{\mu_Y t} e^{\frac{1}{2}\sigma_Y^2 t^2} \\ &= e^{(\mu_X + \mu_Y)t} e^{\frac{1}{2}(\sigma_X^2 + \sigma_Y^2)t^2} \end{aligned}$$

Therefore $X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$

Problem 3. Let $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$ be i.i.d. normal random variables. Let $S_n = X_1 + \cdots + X_n$. Explain why S_n/n is normally distributed and give its mean and variance.

$S_n \sim N(n\mu, n\sigma^2)$ using Problem 2

Moreover $\frac{S_n}{n}$ is also normally distributed since (shifting and) scaling a normal random variable gives a normal random variable. Moreover

$$E\left[\frac{S_n}{n}\right] = \frac{1}{n} E[S_n] = \frac{1}{n}(\mu n) = \mu$$

$$V\left(\frac{S_n}{n}\right) = \frac{1}{n^2} V(S_n) = \frac{1}{n^2}(n\sigma^2) = \frac{\sigma^2}{n}.$$

So $\frac{S_n}{n} \sim N(\mu, \frac{\sigma^2}{n})$

Problem 4. Before paying at a fruit stand, your fruit is weighed on a scale that is a bit unreliable. The weight output by the scale is a random variable $X = w + M$ where w is the true weight of your fruit and $M \sim N(0, 4)$ is the measurement error (in ounces).

a. If you weight your fruit, what is the probability that the weight output by the scale is within 1 ounce of the true weight w?

b. Suppose you weigh your fruit 5 times and take the average of the resulting weights. What is the probability that the average is within 1 ounce of the true weight w?

(a) $P(|X - w| \le 1) = P(|M| \le 1)$

$= P(-1 \le M \le 1)$

$= P(-\frac{1}{2} \le Z \le \frac{1}{2})$

$= 0.383$

(b) Let $X_1, \dots, X_5$ be the result of the 5 weighings

Note $X_1, \dots, X_5 \sim N(w, 4)$ are i.i.d. and

$$\frac{S_5}{5} = \frac{X_1 + \dots + X_5}{5} \sim N(w, \tfrac{4}{5}).$$ So

$$P\left(\left|\frac{S_5}{5} - w\right| \le 1\right) = P\left(-1 \le \frac{S_5}{5} - w \le 1\right)$$

$$= P\left(\frac{-1}{\sqrt{4/5}} \le Z \le \frac{1}{\sqrt{4/5}}\right)$$

$$= P\left(-\frac{\sqrt{5}}{2} \le Z \le \frac{\sqrt{5}}{2}\right)$$ (use pnorm($\sqrt{5}/2$, 0, 1) - pnorm($-\sqrt{5}/2$, 0, 1))

$= 0.7364475$