§ 1.5, 1.6 Counting I

Def Sampling with replacement from a set means we draw an element from the set but put it back in (ie. we replace it) before the next draw.

Sampling without replacement means we don't put it back.

Example Suppose we're making a 4 letter "word" (doesn't have to be real word) by sampling with replacement from the set $\{A, B, C\}$.

① Express the set of all 4-letter words as as set Ω of 4-tuples

② How many 4 letter words are possible?

③ How many palindromes that begin with 'A' are possible?

a word that is the same forward and backward, eg "racecar"

① $\Omega = \{(x_1, x_2, x_3, x_4) : x_i \in \{A, B, C\} \text{ for each } i = 1,2,3,4\}$

② $|\Omega| = 3 \times 3 \times 3 \times 3 = 3^4 = 81$

# of choices for first, second, third, fourth letter

③ $1 \times 3 \times 1 \times 1 = 3$

1st and 4th letter must be 'A'

3rd letter must match 2nd

<u>Multiplication principle</u> Consider an n-element <u>ordered</u> sequence $(a_1, a_2, \ldots, a_n)$.

If there are k_1 possible values of a_1,

k_2 possible values of a_2,

$\vdots$

k_n possible values of a_n,

then there are $k_1 \times k_2 \times \cdots \times k_n$ possible sequences.

<u>Example</u> Nine balls, numbered 1 to 9, are placed in an urn. You draw 3 balls, one at a time. Find the probability the first 2 balls are even numbered if the sampling was done

① with replacement

② without replacement.

Let A be the event that the first 2 balls are even and let Ω be the sample space.

① $\Omega = \{(x_1, x_2, x_3) : x_i \in \{1, \ldots, 9\} \text{ for each } i = 1,2,3\}$

$A = \{(x_1, x_2, x_3) : x_1, x_2 \text{ are even}\}$

$|\Omega| = 9^3$, $|A| = 4 \times 4 \times 9$, $P(A) = \frac{16}{81}$

② $\Omega = \{(x_1, x_2, x_3) : x_i \in \{1, \ldots, 9\} \text{ for each } i = 1,2,3 \text{ and all different}\}$

$|\Omega| = 9 \times 8 \times 7$, $|A| = 4 \times 3 \times 7$, $P(A) = \frac{4 \times 3}{9 \times 8} = \frac{1}{6}$

Def A <u>permutation</u> is an ordering of the elements of a set.

Example Consider the letters A, B, C, D, E, F.

① How many permutations (ie. arrangements) are there of these letters?

② What is the probability a randomly chosen permutation has the letters A and B next to each other?

① $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6!$ "factorial"

# of choices for first, second, third, etc. letter in arrangement

② Let S be the event A and B are next to each other and let Ω be the sample space.

$|\Omega| = 6!$, $|S| = 5! + 5!$ ← # of arrangements, treating 'BA' as one letter

# of arrangements treating 'AB' as one letter

$$P(S) = \frac{2 \times 5!}{6!} = \frac{2}{6} = \frac{1}{3}.$$

Problem 1. Some debit card security pins are made of 4 digits from the numbers 0 to 9 with reptition allowed. Assume a security pin is chosen randomly with all equally likely.

1. Find the number of elements in the sample space.
2. Find the probability of each of the following events.
   (a) The security pin does not contain the number 1.
   (b) The security pin contains at least one 1.
   (c) The security pin contains exactly one 1.

ⓐ $|\Omega| = 10^4$

ⓑ ① $\frac{9^4}{10^4}$ ② $1 - \frac{9^4}{10^4}$ ③ $\frac{4 \times 9^3}{10^4}$

# of choices for which digit is 1.

Problem 2. A standard 52 card deck contains 13 cards of each suit; that is 13 clubs, 13 spades, 13 hearts, and 13 diamonds. Suppose 4 cards are drawn, one at a time, without replacement.

1. Find the number of elements in the sample space.
2. Find the probability of each of the following events.
   (a) All 4 cards are of different suits.
   (b) All 4 cards are of the same suit.

(a) $|\Omega| = 52 \times 51 \times 50 \times 49$

(b) (1) $\dfrac{52 \times 39 \times 26 \times 13}{52 \times 51 \times 50 \times 49} = \dfrac{4! \times 13^4}{52 \times 51 \times 50 \times 4}$

(2) $\dfrac{52 \times 12 \times 11 \times 10}{52 \times 51 \times 50 \times 49}$

Problem 3. A coin is flipped six times. The sample space Ω of this experiment consists of 6-element ordered sequences whose entries are each H or T.

1. Find $|\Omega|$
2. Find the number of elements in each of the following events.
   (a) The first two flips are heads and the last two flips are tails.
   (b) Exactly one of the flips is heads.
   (c) At least one of the flips is heads.

(a) $|\Omega| = 2^6$

(b) (1) 2^2 (2) 6 (3) $2^6 - 1$

Problem 4. Eight people will be seated in a row. Find the number of possible seating arrangements given the following restrictions.

1. No restrictions.
2. Suppose 4 people are wearing glasses and 4 are not. Glasses wearers and non-wearers must be alternated.
3. Suppose 5 people are children and 3 are not. The 5 children must be seated consecutively.
4. Alice and Bob are two of the eight people. They must be seated so that there exactly two people between them.

(a) $8!$

(b) $\underbrace{4! \times 4!}_{\text{\# of arrangements, assuming first seat goes to a glasses-wearer}} + \underbrace{4! \times 4!}_{\text{\# of arrangements, assuming first seat goes to a non-glasses-wearer}}$

(c) $\underbrace{4}_{\text{\# of choices for where the first child sits}} \times \underbrace{5!}_{\text{\# of arrangements of the children}} \times \underbrace{3!}_{\text{\# of arrangements of the non-children}}$

(d) $\underbrace{5}_{\text{\# of choices for first seat}} \times \underbrace{2!}_{\text{\# of arrangements of A and B}} \times \underbrace{6!}_{\text{\# of arrangements of everyone else}}$