§9.1 Conditional Densities

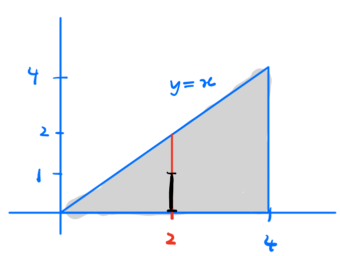
Def If X and Y are continuous random variables with joint density $f(x,y)$, the conditional density of Y, given $X=x$, is given by $f_{Y|X}(y|x) = \dfrac{f(x,y)}{f_X(x)}$

where x is a given constant such that $f_X(x) > 0$.

Example Let X and Y have joint density

$$f(x,y) = \begin{cases} \dfrac{y}{4x} & 0<y<x<4 \\ 0 & \text{otherwise.} \end{cases}$$

Find $P(Y<1 \mid X=2)$.



Our goal is to compute

$$P(0<Y<1 \mid X=2)$$
$$= \int_0^1 f_{Y|X}(y|2)\,dy$$

Steps:

① find $f_X(x)$ and then write a formula for

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$$

② plug in $x=2$ to get

$$f_{Y|X}(y|2) = \frac{f(2,y)}{f_X(2)}$$

(3) integrate.

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(1) $$f_X(x) = \int_{y=0}^{y=x} f(x,y)\,dy$$

$$= \int_0^x \frac{y}{4x}\,dy$$

$$= \frac{1}{8x} y^2 \Big|_0^x$$

$$= \frac{x}{8} \quad \text{when } 0<x<4$$

Therefore $$f_{Y|X}(y|x) = \frac{y/(4x)}{x/8} = \frac{2y}{x^2} \quad \text{when } 0<y<x$$

(2) $$f_{Y|X}(y|2) = \frac{2y}{2^2} = \frac{y}{2} \quad \text{when } 0<y<2$$

(3) $$P(Y<1 \mid X=2) = \int_0^1 \frac{y}{2}\,dy = \frac{1}{4} y^2 \Big|_0^1 = \frac{1}{4}$$

Example Suppose X and Y model lifetimes in minutes of components A and B. Their joint density is

$$f(x,y) = \begin{cases} 2e^{-x}e^{-y} & 0<x<y<\infty \\ 0 & \text{otherwise.} \end{cases}$$

Given that component A fails at exactly 5 minutes find the probability that B fails in less than 15 minutes

We want to compute

$$P(Y \le 15 \mid X=5) = \int_5^{15} f_{Y|X}(y|5)\,dy$$

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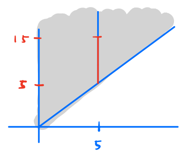


$$f_{X}(x) = \int_{y=x}^{\infty} f(x,y)\,dy$$

$$= \int_{x}^{\infty} 2e^{-x}e^{-y}\,dy$$

$$= -2e^{-x}e^{-y}\Big|_{x}^{\infty}$$

$$= 2e^{-2x} \quad \text{when } x>0$$

Therefore $f_{Y|X}(y|x) = \dfrac{f(x,y)}{f_{X}(x)}$

$$= \frac{2e^{-x}e^{-y}}{2e^{-2x}}$$

$$= e^{x-y} \quad \text{when } y>x$$

2 $f_{Y|X}(y|5) = e^{5-y}$ when $y>5$

3 $$P(Y \le 15 \,|\, X=5) = \int_{5}^{15} e^{5-y}\,dy$$

$$= e^{5}\int_{5}^{15} e^{-y}\,dy$$

$$= -e^{5}e^{-y}\Big|_{5}^{15}$$

$$= 1 - e^{-10}$$

Problem 1. Let X and Y have joint density

$$f(x,y) = \begin{cases} 4e^{-2x} & 0 < y < x < \infty \\ 0 & \text{otherwise.} \end{cases}$$

a. Make a sketch of supp(f).

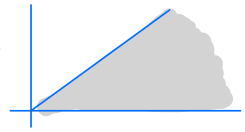
b. Find the marginal density of X.

c. Write an expression for $f_{Y|X}(y \mid x)$.

d. Write an expression for $f_{Y|X}(y \mid 5)$.

e. Find $P(Y > 1 \mid X = 5)$ by computing an integral.

(a)



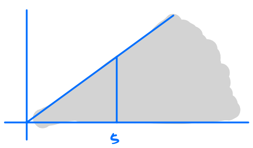
supp (f)

(b) $f_X(x) = \int_{y=0}^{y=x} 4e^{-2x}\, dy = 4xe^{-2x}$ when $x > 0$.

(c) $f_{Y|X}(y|x) = \dfrac{f(x,y)}{f_X(x)} = \dfrac{4e^{-2x}}{4xe^{-2x}} = \dfrac{1}{x}$ when $0 < y < x$

(d) $f_{Y|X}(y|5) = \frac{1}{5}$ when $0 < y < 5$ (so $Y \sim \text{Unif}(0,5)$)

(e)



$$P(Y > 1 \mid X = 5) = \int_1^5 f_{Y|X}(y|5)\, dy$$

$$= \int_1^5 \tfrac{1}{5}\, dy = \tfrac{4}{5}.$$

Problem 2. Suppose you are given the conditional density $f_{Y|X}(y \mid x)$ and marginal densities $f_X(x)$ and $f_Y(y)$. Try to derive a formula for the conditional density $f_{X|Y}(x \mid y)$ in terms of these three given densities. What do you think this formula should be called?

$$f_{X|Y}(x,y) = \frac{f(x,y)}{f_Y(y)} \quad \text{(by definition of conditional density)}$$

$$= \frac{f_{Y|X}(y|x)\, f_X(x)}{f_Y(y)} \quad \left(\text{by rearranging } f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}\right)$$

This is called Bayes' formula.

Problem 3. Suppose Alice picks a random number X uniformly distributed in the interval $(0, 10)$. Then if Alice's number is $X = x$, Bob picks a number Y uniformly distributed in the interval $(0, x)$.

a. Suppose we know Alice picked the number 6. Find the probability that Bob's number is greater than 4 by first doing the following.

   1. State the marginal density $f_X(x)$ of X.
   2. State the conditional density $f_{Y|X}(y \mid x)$ of Y given $X = x$. *Note no calculation is necessary.*

b. Suppose we only saw the second step of the experiment. That is, we saw that Bob picked 1. The find the probability that Alice's number is greater than 9 by first doing the following.

   1. State the joint density of X and Y. *Note no calculation is necessary.*
   2. Find the marginal density $f_Y(y)$ of Y.
   3. Use your answer to Problem 2 to state the conditional density $f_{X|Y}(x \mid 1)$ of X given $Y = 1$.

(a)

1. $$f_X(x) = \begin{cases} \frac{1}{10} & 0<x<10 \\ 0 & \text{else} \end{cases}$$

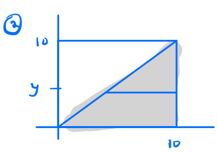
2. $$f_{Y|X}(y \mid x) = \begin{cases} \frac{1}{x} & 0<y<x \\ 0 & \text{else.} \end{cases}$$

Therefore
$$P(Y>4 \mid X=6) = \int_4^\infty f_{Y|X}(y \mid 6)\,dy = \int_4^6 \frac{1}{6}\,dy = \frac{2}{6} = \frac{1}{3}.$$

(b)

1. $$f(x,y) = f_{Y|X}(y \mid x)\, f_X(x) = \begin{cases} \frac{1}{10x} & 0<y<x<10 \\ 0 & \text{else} \end{cases}$$

2.



$$f_Y(y) = \int_{x=y}^{x=10} f(x,y)\,dx = \int_y^{10} \frac{1}{10x}\,dx = \frac{1}{10} \ln x \Big|_y^{10} = \frac{1}{10}(\ln 10 - \ln y)$$

③ $$f_{X|Y}(x|1) = \frac{f_{Y|X}(1|x)\, f_X(x)}{f_Y(1)}$$

$$= \frac{\frac{1}{10x}}{\frac{1}{10}\ln 10}$$

$$= \frac{1}{x\ln 10} \quad \text{when} \quad 1 < x < 10$$

Therefore $$P(X > 9 \mid Y = 1) = \int_9^\infty f_{X|Y}(x|1)\,dx$$

$$= \int_9^{10} \frac{1}{x\ln 10}\,dx$$

$$= \frac{1}{\ln 10}\left(\ln x \Big|_9^{10}\right)$$

$$= \frac{1}{\ln 10}(\ln 10 - \ln 9)$$

$$= 1 - \frac{\ln 9}{\ln 10}$$

<u>Example</u> Suppose $X \sim Unif(0,1)$ represents an unknown parameter of an exponential random variable, which represents the time between hurricanes in a certain region.

That is, conditional on $X = x$, $Y \sim Exp(x)$ represents the time between hurricanes in years.

① Suppose $X = \frac{1}{3}$. Find the conditional probability of having more than 4 years between hurricanes.

$$f_{Y|X}(y|x) = x e^{-xy} \quad \text{when } y > 0 \text{ and}$$

$$f_{Y|X}(y|\tfrac{1}{3}) = \tfrac{1}{3} e^{-\frac{1}{3}y} \quad \text{when } y > 0.$$

Therefore $$P(Y > 4 \mid X = \tfrac{1}{3}) = \int_4^\infty \tfrac{1}{3} e^{-\frac{1}{3}y} \, dy$$

$$= e^{-4/3}$$

② Suppose we know that the time between hurricanes was exactly 2 years. Find the conditional density of X.

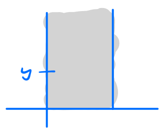
<u>Bayes' formula</u> $$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)}$$

We have $$f_{Y|X}(y|x) = \begin{cases} x e^{-xy} & y > 0 \\ 0 & y \le 0 \end{cases}$$

and

$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{else.} \end{cases}$$

Therefore $f(x,y) = f_{Y|X}(y|x)\, f_X(x)$



$$= \begin{cases} x e^{-xy} & 0 < x < 1,\ y > 0 \\ 0 & \text{else} \end{cases}$$

So $f_Y(y) = \displaystyle\int_{x=0}^{x=1} f(x,y)\,dx$

$$= \int_0^1 x e^{-yx}\,dx \qquad u = x \quad dv = e^{-yx}dx \qquad du = dx \quad v = -\frac{1}{y}e^{-yx}$$

$$= \frac{-x}{y} e^{-yx}\Big|_{x=0}^{x=1} + \int_0^1 \frac{1}{y} e^{-yx}\,dx$$

$$= \frac{-1}{y} e^{-y} - \frac{1}{y^2} e^{-yx}\Big|_{x=0}^{x=1}$$

$$= -\frac{1}{y} e^{-y} - \frac{1}{y^2} e^{-y} + \frac{1}{y^2}$$

Therefore $f_{X|Y}(x|2) = \dfrac{f_{Y|X}(2|x)\, f_X(x)}{f_Y(2)}$

$$= \frac{x e^{-2x}}{-\frac{1}{2}e^{-2} - \frac{1}{4}e^{-2} + \frac{1}{4}}$$

$$= \frac{4e^{-2x}}{1 - 3e^{-2}} \qquad 0 < x < 1.$$

Note this density is maximized at $x = \frac{1}{2}$