§ 9.3 Conditional Expectation

Definition The conditional expectation of Y given $X = x$ is given by

$$E[Y \mid X = x] = \begin{cases} \sum_{y \in S} y \cdot P(Y = y \mid X = x) & \text{when } Y \text{ is discrete} \\ \int_{-\infty}^{\infty} y\, f_{Y|X}(y|x) & \text{when } Y \text{ is continuous.} \end{cases}$$

Example Suppose X and Y have joint pmf given by the following table:

	$X = x_1$	$X = x_2$	
$Y = 1$	0.1	0.3	0.4
$Y = 2$	0.2	0.4	0.6
	0.3	0.7	

Find $E[Y \mid X = x_1]$ and $E[Y \mid X = x_2]$

The conditional pmf of Y given $X = x_1$ is

$$P(Y = k \mid X = x_1) = \frac{P(Y = k, X = x_1)}{P(X = x_1)}$$

$$= \begin{cases} 1/3 & k = 1 \\ 2/3 & k = 2 \end{cases}$$

Therefore $E[Y \mid X = x_1] = 1(1/3) + 2(2/3) = 5/3$

The conditional pmf of Υ given $\Xi = x_2$ is

$$P(\Upsilon = k \mid \Xi = x_2) = \frac{P(\Upsilon = k, \Xi = x_2)}{P(\Xi = x_2)}$$

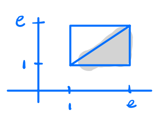
$$= \begin{cases} 3/7 & k = 1 \\ 4/7 & k = 2 \end{cases}$$

Therefore $E[\Upsilon \mid \Xi = x_1] = 1(3/7) + 2(4/7) = 11/7$

Example Suppose Ξ and Υ have joint density

$$f(x,y) = \begin{cases} \frac{2}{xy} & 1 < y < x < e \\ 0 & \text{else.} \end{cases}$$

Find $E[\Upsilon \mid \Xi = x]$ and $E[\Upsilon \mid \Xi = 2]$.



Note $f_{\Xi}(x) = \int_{y=1}^{y=x} \frac{2}{xy}\,dy$

$$= \frac{2}{x}\left(\ln y \Big|_1^x\right) = \frac{2\ln x}{x} \qquad 1 < x < e$$

Therefore $f_{\Upsilon|\Xi}(y|x) = \dfrac{f(x,y)}{f_{\Xi}(x)} = \dfrac{\frac{2}{xy}}{\frac{2\ln x}{x}} = \dfrac{1}{y\ln x} \qquad 1 < y < x$

and so $E[\Upsilon \mid \Xi = x] = \int_{-\infty}^{\infty} y\, f_{\Upsilon|\Xi}(y|x)\,dy$

$$= \int_1^x \frac{y}{y\ln x}\,dy = \frac{x-1}{\ln x},$$

$E[\Upsilon \mid \Xi = 2] = \frac{1}{\ln 2}$.

Problem 1. Consider the random variables X and Y with joint density

$$f(x,y) = \begin{cases} e^{-y} & 0 < x < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

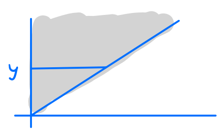
Find the conditional expectation $E[X \mid Y = y]$ through the following steps.

a. Find the marginal density $f_Y(y)$ of Y.

b. Find the conditional density $f_{X|Y}(x \mid y)$, making sure to take note of the interval of values of x for which the conditional density is non-zero. This interval will depend on y.

c. Compute $E[X \mid Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x \mid y)\, dx$. *Note it's possible to avoid doing any computation in this step by thinking intuitively about your answer to the previous part.*

(a)



$$f_Y(y) = \int_{x=0}^{x=y} e^{-y}\, dx$$

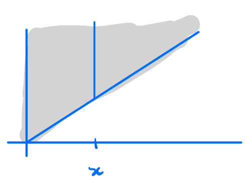
$$= y e^{-y} \quad \text{when } y > 0.$$

(b) $$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{e^{-y}}{y e^{-y}} = \frac{1}{y} \quad \text{when } 0 < x < y.$$

(c) Since $X \mid Y = y \sim \text{Unif}(0, y)$, $E[X \mid Y = y] = \frac{y}{2}$.

Problem 2. Repeat similar steps as the previous problem to compute $E[Y \mid X = x]$.

(a)



$$f_X(x) = \int_{y=x}^{y=\infty} e^{-y}\, dy$$

$$= -e^{-y} \Big|_{x}^{\infty} = e^{-x} \quad \text{when } x > 0$$

(b) $$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{e^{-y}}{e^{-x}} = e^{x-y} \quad \text{when } y > x$$

(c) $$E[Y \mid X = x] = \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y|x)\, dy$$

$$= \int_x^{\infty} y e^{x-y}\, dy \qquad \begin{array}{l} u = y - x \\ du = dy \end{array}$$

$$= \int_0^{\infty} (u + x) e^{-u}\, du = \underbrace{\int_0^{\infty} u e^{-u}\, du}_{\text{expectation of Exp(1)}} + x \int_0^{\infty} \underbrace{e^{-u}}_{\text{density of Exp(1)}} du = 1 + x$$

$$= 1 + x.$$

Problem 3. Suppose $X \sim \text{Unif}(\{1/2, 1/3, 1/4\}$ represents the unknown heads probability of a coin. Let Y be the number of heads that result in tossing this coin 5 times.

a. Give the conditional probability mass function of Y given $X = x$.

b. Find $E[Y \mid X = 1/2], E[Y \mid X = 1/3], E[Y \mid X = 1/4]$. *Note that this can be done with a minimal amount of computation.*

c. State a general formula for $E[Y \mid X = x]$.

d. Make a conjecture for how to compute $E[Y]$ and give its value based on your conjecture.

(a) $P(Y = k) = \binom{5}{k} x^k (1-x)^{5-k}$, $k = 0, 1, 2, 3, 4, 5$.

(b) Since $Y \mid X = x \sim \text{Bin}(5, x)$, $E[Y \mid X = x] = \begin{cases} 5/2 & x = \frac{1}{2} \\ 5/3 & x = \frac{1}{3} \\ 5/4 & x = \frac{1}{4} \end{cases}$

(c) $E[Y \mid X = x] = 5x$

Problem 4. Suppose Alice picks a random number X uniformly distributed in the interval $(0, 10)$. Then if Alice's number is $X = x$, Bob picks a number Y uniformly distributed in the interval $(0, x)$.

a. Give the conditional density of Y given $X = x$.

b. Find a general formula for $E[Y \mid X = x]$. *Note that this can be done with a minimal amount of computation.*

c. Make a conjecture for how to compute $E[Y]$ and give its value based on your conjecture.

(a) $f_{Y|X}(y|x) = \begin{cases} \frac{1}{x} & 0 < y < x \\ 0 & \text{otherwise.} \end{cases}$

(b) Since $Y \mid X = x \sim \text{Unif}(0, x)$, $E[Y \mid X = x] = \frac{x}{2}$.

<u>Example</u> Suppose $X \sim \text{Unif}(\{1/2, 1/3, 1/4\})$ represent heads probability of a possibly biased coin. Let Y be the number of tosses of this coin until first heads. Find $E[Y \mid X = x]$.

$Y \mid X = x \sim \text{Geom}(x)$, So $E[Y \mid X = x] = \frac{1}{x}$.