§ 9.3 Conditional Expectation

Last we discussed $E[Y \mid X = x]$, which we can think of as a function of x. Today we introduce a new notion of conditional expectation, which is a random variable.

Def Given X and Y, jointly distributed random variables (discrete or continuous), the conditional expectation of Y given X is the random variable $E[Y/X]$ derived as a transformation of X:

① Let $g(x) = E[Y / X = x]$

② Define $E[Y \mid X] = g(X)$.

Example Suppose $X \sim \text{Unif}\{1/2, 1/3, 1/4\}$ is the unknown heads probability of a coin. Let Y be the number of heads that result from tossing this coin 5 times.

Find $E[Y \mid X]$.

Since $Y \mid X = x \sim Bin(5, x)$, $E[Y \mid X = x] = 5x$.

Therefore $E[Y \mid X] = 5X$.

Example Suppose Alice picks a random number $X \sim \text{Unif}(0,10)$. Then Bob picks a real number Y uniformly at random between 0 and Alice's number. Find $E[Y|X]$.

Since $Y \mid X = x \sim \text{Unif}(0,x)$, $E[Y \mid X = x] = \frac{x}{2}$.

Therefore $E[Y \mid X] = \frac{X}{2}$.

Theorem (Law of Total Expectation). $E[Y] = E[E[Y|X]]$

Proof (discrete case) Let $g(X) = E[Y|X]$. Then

$$\begin{aligned}
E[E[Y|X]] &= E[g(X)] \\
&= \sum_x g(x) P(X = x) \\
&= \sum_x E[Y \mid X = x] P(X = x) \\
&= \sum_x \left(\sum_y y \cdot P(Y = y \mid X = x) \right) P(X = x) \\
&= \sum_x \sum_y y \cdot P(Y = y, X = x) \\
&= \sum_y y \sum_x P(Y = y, X = x) \\
&= \sum_y y P(Y = y) \\
&= E[Y].
\end{aligned}$$

Example Compute $E[Y]$ in previous two examples.

① $E[Y] = E[E[Y|X]]$

$= E[5X]$

$= 5E[X]$

$= 5\left(\frac{1}{2}\left(\frac{1}{3}\right) + \frac{1}{3}\left(\frac{1}{3}\right) + \frac{1}{4}\left(\frac{1}{3}\right)\right)$

$= \frac{65}{36}$

② $E[Y] = E[E[Y|X]]$

$= E[\frac{1}{2}X]$

$= \frac{1}{2}E[X]$

$= \frac{5}{2}$.

Problem 1. Alice will harvest a random number $T \sim \text{Pois}(55)$ of tomatoes from her vegetable garden over the course of a season, distributed. Suppose that each tomato, independent of others, is defective with probability 0.4. Let X be the number of defective tomatoes.

a. State the conditional distribution of X given $T = n$.

b. Find $E[X \mid T]$.

c. Find $E[X]$.

ⓐ $X \mid T = n \sim Bin(n, 0.4)$

ⓑ $E[X \mid T = n] = 0.4n$, so $E[X \mid T] = 0.4T$

ⓒ $E[X] = E[E[X|T]] = E[0.4T] = 0.4\,E[T] = 0.4(55) = 22$.

Problem 2. Suppose that the number of emails you get in a day is $N \sim \text{Pois}(20)$. Moreover, the time it takes you to read and respond to each is random and independent, from email to email and independent of the number of emails you get. Suppose that the time for each email is exponentially distributed with a mean of 15 minutes. Let $X_1, \ldots, X_N$ be the time spent for each of these N emails, and let T denote the total time in hours spent on emails in a given day.

a. Find $E[T \mid N = n]$.

b. Find $E[T \mid N]$.

c. Find $E[T]$.

(a) $E[T \mid N=n] = E[X_1 + \cdots + X_N \mid N=n] = \frac{1}{4} n.$

(b) $E[T \mid N] = \frac{1}{4} N$

(c) $E[T] = E[E[T \mid N]] = E[\frac{1}{4} N] = \frac{1}{4} E[N] = \frac{1}{4}(20) = 5$

Problem 3. Suppose that X and Y have joint density

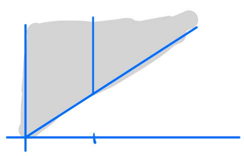
$$f(x,y) = \begin{cases} e^{-y} & 0 < x < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

a. Find $E[Y \mid X = x]$.

b. Find $E[Y \mid X]$.

c. Find $E[Y]$.

(a)



$$f_X(x) = \int_{y=x}^{y=\infty} e^{-y}\, dy$$

$$= -e^{-y}\Big|_x^{\infty} = e^{-x} \quad \text{when } x > 0$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{e^{-y}}{e^{-x}} = e^{x-y} \quad \text{when } y > x$$

$$E[Y \mid X = x] = \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y|x)\, dy$$

$$= \int_x^{\infty} y e^{x-y}\, dy \qquad u = y - x, \quad du = dy$$

$$= \int_0^{\infty} (u+x) e^{-u}\, du$$

$$= 1 + x.$$

(b) $E[Y \mid X] = 1 + X$

(c) $E[Y] = E[1 + X] = 1 + E[X] = 2$