§ 10.1 - 10.2, 10.4 Law of Large Numbers and Monte Carlo Integration.

Our goal today is to study the behavior of sample averages of i.i.d. sequences: let $X_1, X_2, \ldots$ be an i.i.d sequence with $\mu = E[X_i]$, $\sigma^2 = V(X_i)$ (not necessarily normally distributed, any distribution!) and let $\frac{S_n}{n} = \frac{X_1 + \cdots + X_n}{n}$ be the sample mean.

Notice ① $E[\frac{S_n}{n}] = E[\frac{X_1 + \cdots + X_n}{n}]$

$$= \frac{1}{n}(E[X_1] + \cdots + E[X_n]) = \mu.$$

② $V(\frac{S_n}{n}) = V(\frac{X_1 + \cdots + X_n}{n})$

$$= \frac{1}{n^2}(V(X_1) + \cdots + V(X_n)) = \frac{\sigma^2}{n}.$$

So $\lim_{n \to \infty} V(\frac{S_n}{n}) = 0$

Intuition: Since $V(\frac{S_n}{n}) \to 0$ as $n \to \infty$, the random variable $\frac{S_n}{n}$ is converging to a constant (μ) as $n \to \infty$.

Theorem (Strong Law of Large Numbers, Kolmogorov, 1930)

$$P\left(\lim_{n \to \infty} \frac{S_n}{n} = \mu \right) = 1.$$

In other words, using n i.i.d. samples of a distribution, as $n \to \infty$, the sample mean is guaranteed to converge to the mean of the distribution.

Remarks

① When talking about sequences of random variables there are different notions or "modes" of convergence. The one in the SLLN is called almost sure convergence.

② There is a theorem called the Weak Law of Large Numbers which proves $\frac{S_n}{n} \to \mu$ under a different mode of convergence called convergence in probability (which is a form of convergence which is "weaker" than or implied by almost sure convergence).

③ Pages 414-416 give a nice discussion about this for those with experience in real analysis.

Example If $X_1, X_2, \ldots \sim Exp(5)$ are i.i.d. then

$$\frac{S_n}{n} = \frac{X_1 + X_2 + \cdots + X_n}{n} \approx E[X_1] = \frac{1}{5}$$

(and this approximation improves as $n \to \infty$).

Monte Carlo Techniques We can use repeated sampling to do numerical approximation of thing like integrals.

Goal Compute $\int_0^1 g(x)\,dx$ where $g(x)$ is a given function.

Algorithm ① Generate $X_1, \ldots, X_n \sim \text{Unif}(0,1)$ i.i.d.

② Compute $\dfrac{g(X_1) + \cdots + g(X_n)}{n}$

Conclusion By SLLN, $\dfrac{g(X_1) + \cdots + g(X_n)}{n} \approx E[g(X_1)]$

$$= \int_0^1 g(x) \cdot \underset{\substack{\uparrow \\ \text{density of } X_1}}{1}\, dx.$$

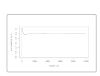
Example $\int_0^1 (\sin x)^{\cos x}\, dx.$

① Generate $X_1, \ldots, X_n \sim \text{Unif}(0,1)$ i.i.d.

② Compute $\dfrac{(\sin(X_1))^{\cos(X_1)} + \cdots + (\sin(X_n))^{\cos(X_n)}}{n}$

````
```{r}
library(dplyr)
n = 1e4 # sample size
X = runif(n,0,1)
g = function(x) sin(x)^cos(x)
mean(g(X)) # Monte Carlo approximation
integrate(g,0,1) # R's built-in integrator
plot(1:n, cummean(g(X)), type = 'l', xlab = "sample size", ylab = "approximation value")
```
````

R Console



```
[1] 0.5000953
0.5013249 with absolute error < 1.1e-09
```

Goal Compute $\int_0^\infty g(x)\,dx$ where $g(x)$ is a given function.

Algorithm ① Generate $X_1, \dots, X_n \sim Exp(1)$ i.i.d.

② Compute $\dfrac{g(X_1)e^{X_1} + \cdots + g(X_n)e^{X_n}}{n}$

Conclusion By SLLN,

$$\frac{g(X_1)e^{X_1} + \cdots + g(X_n)e^{X_n}}{n} \approx E[g(X_1)e^{X_1}]$$

$$= \int_0^\infty g(x)e^x \cdot \underbrace{e^{-x}}_{\text{density of } X_1} dx$$

$$= \int_0^\infty g(x)\,dx.$$

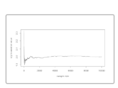
Example Compute $\int_0^\infty x^{-x}\,dx$.

① Generate $X_1, \dots, X_n \sim Exp(1)$ i.i.d.

② Compute $\dfrac{X_1^{-X_1} \cdot e^{X_1} + \cdots + X_n^{-X_n} e^{X_n}}{n}$

```{r}
n = 1e4
X = rexp(n,1)
g = function(x) x^(-x)
mean(g(X)*exp(X))
integrate(g,0,Inf)
plot(1:n, cummean(g(X)*exp(X)), type = 'l', xlab = "sample size", ylab = "approximation value")
```

R Console



```
[1] 1.999112
1.995456 with absolute error < 0.00016
```

See http://tchumley.mtholyoke.edu/m342/ws/day36.Rmd for code for worksheet problems.