§10.6 Proof of the CLT

Today we'll work on proving the Central Limit Theorem.

There are two technical results we'll discuss first.

Def A limit is of indeterminate form if it has one of following informal expressions: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, 0^0, 1^∞, ∞^0, $\infty - \infty$

Theorem (L'Hôpital's Rule) If f and g are differentiable functions such that $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ or

$\lim_{x \to a} f(x) = \pm \infty = \lim_{x \to a} g(x)$ and $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

Example Compute ① $\lim_{x \to 1} \frac{\ln x}{x-1}$ ② $\lim_{x \to \infty} \frac{e^x}{x^2}$ ③ $\lim_{x \to \infty} x \sin\left(\frac{1}{x}\right)$.

① $\lim_{x \to 1} \frac{\ln x}{x-1} \overset{L'H}{=} \lim_{x \to 1} \frac{1/x}{1} = 1$

② $\lim_{x \to \infty} \frac{e^x}{x^2} \overset{L'H}{=} \lim_{x \to \infty} \frac{e^x}{2x} \overset{L'H}{=} \lim_{x \to \infty} \frac{e^x}{2} = \infty$

③ $\lim_{x \to \infty} x \cdot \sin\left(\frac{1}{x}\right) = \lim_{x \to \infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \overset{L'H}{=} \lim_{x \to \infty} \frac{\cos\left(\frac{1}{x}\right) \cdot \frac{-1}{x^2}}{\frac{-1}{x^2}} = 1$

Theorem (Lévy Continuity Theorem)

Suppose $X_1, X_2, \dots$ is a sequence of random variables with moment generating functions $m_1(t), m_2(t), \dots$ and let X be a random variable with mgf $m_X(t)$.

If $\lim_{n\to\infty} m_n(t) = m_X(t)$ for all t, then the sequence $(X_n)_{n\geq 1}$ converges in distribution to X. That is

$\lim_{n\to\infty} P(X_n \leq x) = P(X \leq x)$ (the CDF's of X_n converge to the CDF of X). In other words, to prove convergence in distribution it suffices to prove convergence of mgf's.

Intuition The mgf of a random variable characterize all of its moment which in turn characterize the (shape of the) distribution and its CDF.

Corollary (CLT) Let $X_1, X_2, \ldots$ be an i.i.d. sequence of random variables with finite mean $\mu = E[X_i]$ and variance $\sigma^2 = E[X_i]$ and let $S_n = X_1 + \cdots + X_n$ for each $n \geq 1$. Then $\dfrac{\frac{S_n}{n} - \mu}{\sigma/\sqrt{n}}$ converges in distribution to $Z \sim N(0,1)$. That is,

$$\lim_{n \to \infty} P\left(\frac{\frac{S_n}{n} - \mu}{\sigma/\sqrt{n}} \leq x \right) = P(Z \leq x) \quad \text{for all } x \in \mathbb{R}.$$

Sketch of proof

Let $m_n(t)$ be the mgf of $\dfrac{\frac{S_n}{n} - \mu}{\sigma/\sqrt{n}}$ for each $n \geq 1$ and prove that $m_n(t)$ converges to the mgf of $Z \sim N(0,1)$. Do this in two cases:

1. the case when $\mu = 0$, $\sigma^2 = 1$
2. the case of general μ and σ^2.

A few important reminders about mgf's

1. $m_Z(t) = e^{t^2/2}$ when $Z \sim N(0,1)$
2. $m_X^{(k)}(0) = E[X^k]$
3. $m_{X+Y}(t) = m_X(t)^2$ when X, Y are i.i.d.
4. $m_{aX}(t) = m_X(at)$ for any constant $a \neq 0$.