§ 1.7 Counting II

Example Consider the 7-element set $\{A, B, C, D, E, F, G\}$.

① How many 4-letter words can we make, assuming no repeated letters? (ie we're making an ordered sequence, sampling without replacement)

$$7 \times 6 \times 5 \times 4 = \frac{7!}{3!}$$

② How many 4-element subsets are there of the 7-element set? (With subsets, the order of the elements does not matter)

$$\#\text{ of 4-letter ordered sequences with no repitition} = \left(\#\text{ of 4-letter subsets}\right) \times \#\text{ of permutations of the 4 letters}$$

$$\#\text{ of 4-letter subsets} = \frac{\#\text{ of 4-letter ordered sequences with no repitition}}{\#\text{ of permutations of the 4 letters}} = \frac{\frac{7!}{3!}}{4!} = \frac{7!}{3!4!}$$

General Principle Given an n-element set,

① The number of k-element ordered sequences with no repitition is $\dfrac{n!}{(n-k)!}$

② The number of k-element unordered subsets is $\dfrac{n!}{(n-k)!k!} = \binom{n}{k}$ "n choose k"

called a "binomial coefficient"

Example 25 people participate in a clinical trial.

15 will receive treatment

10 will receive placebo.

Consider a group of 6 people in the trial chosen at random. What is the probability 4 received treatment (and 2 received the placebo)?

$$\frac{\#\text{ of groups of 6 where 4 receive treatment}}{\#\text{ of groups of 6}} = \frac{\binom{15}{4}\binom{10}{2}}{\binom{25}{6}}$$

Example 5 cards are chosen from a 52 card deck. What is the probability of getting 1) 3 K's, 2 Q's ?

2) 3 K's ?

1) $\dfrac{\binom{4}{3}\binom{4}{2}}{\binom{52}{5}}$ 2) $\dfrac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}}$

Example A person has 8 friends, 5 will be invited to a party.

Suppose Alice and Bob are two of the friends.

What is the prob. A and B are both invited?

$$\frac{\binom{2}{2}\binom{6}{3}}{\binom{8}{5}}$$

What is the prob. A is invited but B is not?

$$\frac{\binom{1}{1}\binom{1}{0}\binom{6}{4}}{\binom{8}{5}}$$

Problem 1. An urn contains 20 balls; 10 are green, 6 are white, and 4 are red. Suppose we draw 7 balls at random, without replacement. Find the probability of each of the following events.

a. We choose exactly 4 green balls.

b. We choose 4 green balls and 3 white balls.

c. We choose 2 green, 3 white, and 2 red balls.

ⓐ $\dfrac{\binom{10}{4}\binom{10}{3}}{\binom{20}{7}}$ ⓑ $\dfrac{\binom{10}{4}\binom{6}{3}}{\binom{20}{7}}$ ⓒ $\dfrac{\binom{10}{2}\binom{6}{3}\binom{4}{2}}{\binom{20}{7}}$

Problem 2. Consider tossing a coin 6 times. Use binomial coefficients to express the number of outcomes in each of the following events.

a. Exactly 3 of the flips is heads.

b. Exactly 2 of the flips is heads.

c. Exactly 1 of the flips is heads.

d. There are no heads.

ⓐ $\binom{6}{3}$ ⓑ $\binom{6}{2}$ ⓒ $\binom{6}{1}$ ⓓ $\binom{6}{0}$

Problem 3. We choose 5 cards from the standard deck of cards. Find the probability of each of the following events.

a. Our draw contains exactly one ace. *Note that a deck of 52 cards contains 4 aces.*

b. Our draw contains at least one ace.

c. Our draw contains exactly 2 aces.

ⓐ $\dfrac{\binom{4}{1}\binom{48}{4}}{\binom{52}{5}}$ ⓑ $1-\dfrac{\binom{4}{0}\binom{48}{5}}{\binom{52}{5}}$ ⓒ $\dfrac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}}$

Problem 4. A committee of 7 is to be chosen at random from a group of 5 Republicans, 6 Democrats, and 4 Independents.Find the probability that the committee contains:

a. 2 Republicans, 2 Democrats, and 3 Independents.

b. at least 2 independents.

ⓐ $\dfrac{\binom{5}{2}\binom{6}{2}\binom{4}{3}}{\binom{15}{7}}$ ⓑ $\dfrac{\binom{4}{2}\binom{11}{5}+\binom{4}{3}\binom{11}{4}+\binom{4}{4}\binom{11}{3}}{\binom{15}{7}}$

Problem 5. Alice, Bob, and 7 friends will go on a canoe trip. They'll have 3 canoes that hold 2, 3 and 4 people respectively, and will divide themselves randomly with every arrangement equally likely. What is the probability Alice and Bob are on the same canoe? *(Can you break this down by thinking about the probability they're on the same 4 person canoe, the same 3 person canoe, and the same 2 person canoe separately?)*

$$\frac{\text{\# of ways A,B are on 4 person canoe together} + \text{\# of ways A,B on 3 person canoe together} + \text{\# of ways A,B are on 2 person canoe together}}{\text{\# of ways of putting people on the canoes.}}$$

\# of choices for 4-person canoe

\# of choices for 3 person canoe

\# of choices for 2 person canoe

$$= \frac{\binom{7}{2}\binom{5}{3}\binom{2}{2} + \binom{7}{4}\binom{3}{1}\binom{2}{2} + \binom{7}{4}\binom{3}{3}}{\binom{9}{4}\binom{5}{3}\binom{2}{2}}$$