§1.8 Problem Solving Strategies (complements, inclusion-exclusion)

Theorem (inclusion-exclusion) Let $A, B \subseteq \Omega$ be events.

Then $P(A \cup B) = P(A) + P(B) - P(AB)$

Proof See day 2 lecture notes

Theorem (inclusion-exclusion) Let $A, B, C \subseteq \Omega$ be events.

Then
$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$$

Proof Observe that

$$P(A \cup B \cup C) = P(A \cup B) + P(C) - P((A \cup B)C)$$
$$= P(A) + P(B) - P(AB) + P(C) - P((A \cup B)C).$$

Next note that $(A \cup B)C = AC \cup BC$. Therefore

$$P((A \cup B)C) = P(AC \cup BC)$$
$$= P(AC) + P(BC) - P((AC) \cap (BC))$$
$$= P(AC) + P(BC) - P(ABC).$$

Example 18 students live together in a dorm

7 of them are in a math class

10 in bio

10 in CS

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3 of them in a math class *and* a bio class

4 in math *and* CS

5 in bio *and* CS

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1 in a math class *and* a bio class *and* a CS class

Find the probability a randomly chosen student is in neither a math nor bio nor CS class.

Let M be the event a student is in a math class

B the event they're in a bio class,

C the event they're in a CS class.

Then $P(M) = \frac{7}{18}$, $P(B) = \frac{10}{18}$, $P(C) = \frac{10}{18}$

$P(MB) = \frac{3}{18}$, $P(MC) = \frac{4}{18}$, $P(BC) = \frac{5}{18}$,

$P(MBC) = \frac{1}{18}$.

$$\begin{aligned} P((M \cup B \cup C)^c) &= 1 - P(M \cup B \cup C) \\ &= 1 - \Big[P(M) + P(B) + P(C) \\ &\qquad - P(MB) - P(MC) - P(BC) \\ &\qquad + P(MBC) \Big] \\ &= 1 - \frac{16}{18} = \frac{1}{9}. \end{aligned}$$

Example We roll 4 dice. Find the probability of getting at least one 6.

Let A be the event of rolling at least one 6.

Method 1

Then A^c is the event of rolling no 6.

Let Ω be the sample space.

Then $P(A) = 1 - P(A^c) = 1 - \frac{|A^c|}{|\Omega|} = 1 - \frac{5^4}{6^4}$

Method 2

Let X be the number of 6's rolled. Then

$$P(A) = P(X=1 \text{ or } X=2 \text{ or } X=3 \text{ or } X=4)$$
$$= P(X=1) + P(X=2) + P(X=3) + P(X=4)$$

Method 3 Let A_1, A_2, A_3, A_4 be the events of getting a 6 on roll 1, 2, 3, 4 respectively. (These are not mutually exclusive). Then

$P(A) = P(A_1 \cup A_2 \cup A_3 \cup A_4)$ requires inclusion-exclusion formula for the union of 4 events.

General Rule When computing probabilities of "at least one" events, consider whether the probability of the complementary event is easier.

Problem 1. Consider the random experiment of rolling a die 5 times. Find the probability of each of the following events.

a. All 5 rolls land on the same value.

b. All 5 rolls land on different values.

c. At least 1 of the rolls lands on 4.

d. At least 2 of the rolls land on 4.

(a) $\frac{6}{6^5}$ (b) $\frac{6!}{6^5}$

(c) $1 - \frac{5^5}{6^5}$ (d) $1 - \frac{5^5}{6^5} - \frac{\binom{5}{1}5^4}{6^5}$

Problem 2. A certain town has 3 newspapers: A, B, and C. The proportions of townspeople who read these papers are as follows:

- A: 10 percent, B: 30 percent, C: 5 percent
- A and B: 8 percent, A and C: 2 percent, B and C: 4 percent
- all 3: 1 percent

A person is chosen at random, everyone equally likely. Find the probability that they read

a. at least one newspaper.

b. no newspaper.

c. at least two newspapers.

d. exactly one newspaper.

Let A, B, C be the events that a person likes newspapers A, B, C respectively.

(a) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$

$= 0.10 + 0.30 + 0.05 - 0.08 - 0.02 - 0.04 + 0.01$

$= 0.32$

(b) $P((A \cup B \cup C)^c) = 1 - 0.32 = 0.68$

(c) $P(AB \cup AC \cup BC) = P(AB) + P(AC) + P(BC)$
$- P(ABC) - P(ABC) - P(ABC) + P(ABC)$

$= 0.08 + 0.02 + 0.04 - 0.02$

$= 0.12$

(d) $P((A \cup B \cup C)(AB \cup AC \cup BC)^c) = 0.32 - 0.12 = 0.20$

Problem 3. Consider the experiment of rolling a die 5 times with sample space Ω. Let C be the event that at least one of the 5 rolls lands on 4. For each $k = 1, \ldots, 5$, let

- A_k be the event that roll k lands on 4.
- B_k be the event that among the 5 rolls, exactly k of them land on 4.

Let B_0 be the event no rolls land on 4. State whether each of the following is true or false.

a. $B_0 = C^c$ T

b. $\Omega = B_0 \cup B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5$ T

c. $C = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$ T

d. $A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 = B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5$ T

e. $P(C) = P(A_1) + P(A_2) + P(A_3) + P(A_4) + P(A_5)$ F

f. $P(C) = P(B_1) + P(B_2) + P(B_3) + P(B_4) + P(B_5)$ T