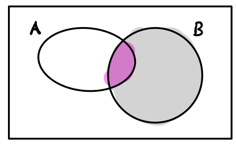
§ 2.1 - 2.3 Introduction to conditional probability

Def Let $A, B \subseteq \Omega$ be given events with $P(B) \neq 0$.
Then the conditional probability of A given B
is $$P(A \mid B) = \frac{P(AB)}{P(B)}$$



We can think of this as reducing the sample space to B and finding the relative size of A within B.

Example In a certain town's population, 30% are zombies, 40% dislike sunrise, and 60% are either a zombie or dislike sunrise or both.

① We choose someone at random from the town. What is the probability they're a zombie and dislike sunrise?

Let Z be the event a randomly chosen resident is a zombie, and let D be the event a randomly chosen resident dislikes sunrise.

$$P(ZD) = P(Z) + P(D) - P(Z \cup D)$$
$$= 0.3 + 0.4 - 0.6 = 0.1$$

② We choose a zombie at random. What is the probability they dislike sunrise?

$$P(D \mid Z) = \frac{P(ZD)}{P(Z)} = \frac{0.1}{0.3} = \frac{1}{3}$$

Example Consider the experiment of rolling a red die and a blue die and suppose you get a sum of 6. Find the probability the red die landed on 5.

Let R and B be the value of the red and blue die respectively.

comma signifies "and"

$$P(R=5 \mid R+B=6) = \frac{P(R=5, R+B=6)}{P(R+B=6)}$$

$$= \frac{P(R=5, B=1)}{P(R+B=6)} = \frac{1/36}{5/36} = \frac{1}{5}$$

Example An urn has 3 red and 4 green balls. You draw 2 balls one at a time, without replacement. Find the probability of getting

1. 2 red balls
2. a red ball and a green ball

Let X_k be the color of the ball on draw k.

Then ① $P(X_1 = R, X_2 = R) = P(X_2 = R \mid X_1 = R)P(X_1 = R)$

$$= \left(\frac{2}{6}\right)\left(\frac{3}{7}\right) = \frac{1}{7}$$

② $P(X_1 = R, X_2 = G) + P(X_1 = G, X_2 = R)$

$$= P(X_2 = G \mid X_1 = R)P(X_1 = R) + P(X_2 = R \mid X_1 = G)P(X_1 = G)$$

$$= \left(\frac{4}{6}\right)\left(\frac{3}{7}\right) + \left(\frac{3}{6}\right)\left(\frac{4}{7}\right) = \frac{4}{7}$$

Example We draw 3 cards from a standard deck of playing cards, one a time without replacement. Find the probability of drawing 3 K's.

Let K_i be the event of drawing a king on draw i.

Then $P(K_3K_2K_1) = P(K_3 | K_2K_1) P(K_2K_1)$

$$= P(K_3|K_2K_1) P(K_2|K_1) P(K_1)$$

$$= \left(\frac{2}{50}\right)\left(\frac{3}{51}\right)\left(\frac{4}{52}\right)$$

General Principle $P(AB) = P(A|B)P(B)$ is useful for "and" probabilities, particularly in random experiments with a sequential structure.

Problem 1. In a certain town 30% of people own cats. Of the cat owners, 20% own dogs, while town-wide, 40% of people own dogs. What is the probability a dog owner owns a cat? What is the probability a person owns both a cat and a dog?

$P(C) = 0.3, \ P(D|C) = 0.2, \ P(D) = 0.4$

① $P(CD) = P(D|C)P(C) = (0.2)(0.3) = 0.06$

② $P(C|D) = \dfrac{P(CD)}{P(D)} = \dfrac{0.06}{0.4} = \dfrac{3}{20} = 0.15$

Problem 2. An urn contains 6 red and 9 green balls. If 4 balls are to be randomly selected without replacement, what is the probability that the first 2 selected are white and the last 2 green?

Let X_k denote draw k.

$P(X_1 = w, X_2 = w, X_3 = g, X_4 = g)$

$$= P(X_1 = w) P(X_2 = w | X_1 = w) P(X_3 = g | X_2 = w, X_1 = w) P(X_4 = g | X_3 = g, X_2 = w, X_1 = w) = \left(\frac{6}{15}\right)\left(\frac{5}{14}\right)\left(\frac{9}{13}\right)\left(\frac{8}{12}\right)$$

Problem 3. A bag contains 15 tiles from the game Scrabble; 5 have the letter M, 5 have the letter N, and 5 have the letter O. Four tiles are chosen at random, one at a time, without replacement. What is the probability that they spell out M-O-M-O (in that order)?

$$\left(\frac{5}{15}\right)\left(\frac{5}{14}\right)\left(\frac{4}{13}\right)\left(\frac{4}{12}\right)$$

Problem 4. For each of the following conditions, find a simplified expression for $P(A \mid B)$.

a. $A = B$

b. $B \subseteq A$

c. $A \subseteq B$

d. A and B are mutually exclusive events.

(a) $P(A \mid B) = \frac{P(AB)}{P(B)} = \frac{P(B)}{P(B)} = 1$

(b) $P(A \mid B) = \frac{P(AB)}{P(B)} = \frac{P(B)}{P(B)} = 1$

(c) $P(A \mid B) = \frac{P(AB)}{P(B)} = \frac{P(A)}{P(B)}$

(d) $P(A \mid B) = \frac{P(AB)}{P(B)} = \frac{P(\phi)}{P(B)} = \frac{1 - P(\Omega)}{P(B)} = 0$

Problem 5. Derive the following formula by simplifying the left hand side until you get the right hand side:

$$P(A \mid B) + P(A^c \mid B) = 1.$$

$$P(A \mid B) + P(A^c \mid B) = \frac{P(AB)}{P(B)} + \frac{P(A^cB)}{P(B)}$$

$$= \frac{P(AB) + P(A^cB)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

Problem 6. A recent college graduate is planning to take the first three actuarial examinations in the coming summer. She will take the first actuarial exam in June. If she passes that exam, then she will take the second exam in July, and if she also passes that one, then she will take the third exam in September. If she fails an exam, then she is not allowed to take any others. The probability that she passes the first exam is 0.9. If she passes the first exam, then the conditional probability that she passes the second one is 0.8, and if she passes both the first and the second exams, then the conditional probability that she passes the third exam is 0.7.

a. What is the probability that she passes all three exams?

b. Given that she did not pass all three exams, what is the conditional probability that she failed the second exam?

Let S_1, S_2, S_3 be the events she passes exams 1, 2, 3, respectively. Then

$$P(S_1) = 0.9, \quad P(S_2 \mid S_1) = 0.8, \quad P(S_3 \mid S_1 S_2) = 0.7$$

① $P(S_1 S_2 S_3) = (0.9)(0.8)(0.7) = 0.504$

② $P(S_2^c \mid (S_1 S_2 S_3)^c) = \dfrac{P(S_2^c (S_1 S_2 S_3)^c)}{P((S_1 S_2 S_3)^c)}$

$$\begin{aligned} S_2^c (S_1 S_2 S_3)^c &= S_2^c (S_1^c \cup S_2^c \cup S_3^c) \quad \text{(DeMorgan)} \\ &= (S_2^c S_1^c) \cup (S_2^c S_2^c) \cup (S_2^c S_3^c) \\ &= \emptyset \cup S_2^c \cup \emptyset \quad \text{(since she can't take exam 2 without passing exam 1, similarly with exams 2 and 3)} \\ &= S_2^c \\ &= S_2^c S_1 \quad (\text{since } S_2^c \subseteq S_1) \end{aligned}$$

So $\dfrac{P(S_2^c (S_1 S_2 S_3)^c)}{P((S_1 S_2 S_3)^c)} = \dfrac{P(S_2^c \mid S_1) P(S_1)}{P((S_1 S_2 S_3)^c)}$

$$= \frac{(1-0.8)(0.9)}{1-0.504}$$