§ 2.4 - 2.5 Law of Total Probability and Bayes' Formula

Theorem (Law of Total Probability)

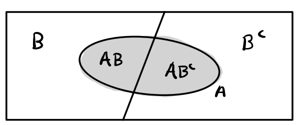
Let $A \subseteq \Omega$ be a given event.

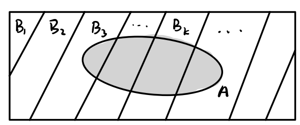
Suppose $B \subseteq \Omega$ is such that $P(B) \neq 0$.

Then
$$P(A) = P(AB) + P(AB^c)$$
$$= P(A|B)P(B) + P(A|B^c)P(B^c)$$

More generally, if $B_1 \cup \cdots \cup B_n = \Omega$ is a disjoint partition of the sample space with $P(B_i) \neq 0$ for all i,

$$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i).$$





Example A blood test is 95% effective in detecting a disease when the disease is actually present. However the test yields a 1% false positive rate among healthy people tested. Suppose 0.5% of the population has the disease.

① What is the probability a randomly selected person is tested and yields a positive result?

Let A be event test returns a positive result.

Let B be the event selected person has the disease.

Then $P(A|B) = 0.95$, $P(A|B^c) = 0.01$, $P(B) = 0.005$

and so $P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$

$$= (0.95)(0.005) + (0.01)(0.995)$$

② If someone gets a positive result, what is the probability they have the disease?

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(A|B)P(B)}{P(A)}$$

$$= \frac{(0.95)(0.005)}{(0.95)(0.005) + (0.01)(0.995)}$$

Bayes' Formula $\quad P(A|B) = \dfrac{P(B|A)P(A)}{P(B)}$

- Useful for "inverting" the conditional probability (i.e. you know $P(B|A)$ but want to find $P(A|B)$).
- Often the denominator is computed using the Law of Total Probability.

Problem 1. Amy has two bags of candy. The first bag contains two packs of M&Ms and three packs of Gummi Bears. The second bag contains four packs of M&Ms and two packs of Gummi Bears. Amy chooses a bag at random—the first chosen with probability 1/4 and the second bag chosen with probability 3/4— and then picks a pack of candy. What is the probability that the pack chosen is Gummi Bears?

Let A_i be event that bag i is chosen.

Let G be event of choosing Gummi Bears

Then $$P(G) = P(G|A_1)P(A_1) + P(G|A_2)P(A_2)$$

$$= \left(\frac{3}{5}\right)\left(\frac{1}{4}\right) + \left(\frac{2}{6}\right)\left(\frac{3}{4}\right)$$

$$= \frac{3}{20} + \frac{1}{4} = \frac{8}{20}$$

Problem 2. Urn A has 5 white and 7 black balls. Urn B has 3 white and 12 black balls. We flip a fair coin. If the outcome is heads, then a ball from urn A is selected, whereas if the outcome is tails, then a ball from urn B is selected. Suppose that a white ball is selected. Given this, what is the probability that the coin landed tails?

Let H, T be events of getting heads and tails.

Let W be event of selecting a white ball.

$$P(T|W) = \frac{P(W|T)P(T)}{P(W)}$$

$$= \frac{P(W|T)P(T)}{P(W|T)P(T) + P(W|H)P(H)}$$

$$= \frac{(3/15)(1/2)}{(3/15)(1/2) + (5/12)(1/2)}$$

Problem 3. Alice and Bob hid a present for their grandmother. With probability 0.6, the present was hidden by Alice; with probability 0.4, it was hidden by Bob. When Alice hides a present, she hides it upstairs 70 percent of the time and downstairs 30 percent of the time. Bob is equally likely to hide it upstairs or downstairs.

a. What is the probability that the present is upstairs?

b. Given that it is downstairs, what is the probability it was hidden by Bob?

Let U be the event the present is upstairs.

Let A, B be the events Alice, Bob hide the present

(a) $P(U) = P(U|A)P(A) + P(U|B)P(B)$

$= (0.7)(0.6) + (0.5)(0.4)$

$= 0.42 + 0.20 = 0.62$

(b) $P(A|U^c) = \dfrac{P(U^c|A)P(A)}{P(U^c)}$

$= \dfrac{(0.3)(0.6)}{0.38} = \dfrac{18}{38} = \dfrac{9}{19}$

Problem 4. A lie-detector test, also called a polygraph, is often given when hiring employees for sensitive positions, but some studies have shown there are issues with their use. According to a 1987 study,

- there is an 88% chance of a positive reading (meaning the test says the subject is lying) when the subject is lying,
- there is an 86% chance of a negative reading (meaning the test says the subject is not lying) when the subject is not lying.

Suppose that on a certain question, there is a 99% chance that the subject is not lying. If the test gives a positive reading, what is the conditional probability that the test is incorrect and the subject is not lying?

Let S be the event of a positive reading and let L be the event the subject is lying.

Then $P(S|L) = 0.88$, $P(S^c|L^c) = 0.86$, $P(L^c) = 0.99$

$P(L^c|S) = \dfrac{P(S|L^c)P(L^c)}{P(S)}$

$= \dfrac{P(S|L^c)P(L^c)}{P(S|L^c)P(L^c) + P(S|L)P(L)}$

$= \dfrac{(0.14)(0.99)}{(0.14)(0.99) + (0.88)(0.01)}$

Problem 5. A study of automobile accidents produced the data in the table below. Suppose an automobile from one of the model years 2012, 2013, or 2014 was involved in an accident. Find the probability that it was from model year 2014.

Model year	Proportion of all vehicles	Probability of involvement in an accident
2014	0.16	0.05
2013	0.18	0.02
2012	0.20	0.03
other	0.46	0.04

Let A_i be the event an automobile from model year $201i$ has an accident

Let B_i be the event a randomly selected automobile is from model year $201i$.

Notice $A_i \subseteq B_i$ for each i.

Then $P(A_4 / B_4) = 0.05$, $P(A_3 | B_3) = 0.02$, $P(A_2 | B_2) = 0.03$

$P(B_4) = 0.16$, $P(B_3) = 0.18$, $P(B_2) = 0.20$

$$P(B_4 | (A_2 \cup A_3 \cup A_4)) = \frac{P(B_4 (A_2 \cup A_3 \cup A_4))}{P(A_2 \cup A_3 \cup A_4)}$$

$$= \frac{P(A_4 | B_4) P(B_4)}{P(A_2 \cup A_3 \cup A_4)}$$

$$= \frac{P(A_4 | B_4) P(B_4)}{P(A_2 \cup A_3 \cup A_4 | B_4) P(B_4) + P(A_2 \cup A_3 \cup A_4 | B_3) P(B_3) + P(A_2 \cup A_3 \cup A_4 | B_2) P(B_2) + P(A_2 \cup A_3 \cup A_4 | \underset{\text{other}}{O}) P(O)}$$

$$= \frac{P(A_4 | B_4) P(B_4)}{P(A_4 | B_4) P(B_4) + P(A_3 | B_3) P(B_3) + P(A_2 | B_2) P(B_2)}$$

$$= \frac{(0.05)(0.16)}{(0.05)(0.16) + (0.02)(0.18) + (0.03)(0.20)}$$