§ 2.6, 3.1 – 3.3 Independence, introduction to random variables

Def Events $A, B \subseteq \Omega$ are called independent if

$$P(A \mid B) = P(A).$$

They are called dependent if they're not independent.

Example Consider drawing a card from a standard deck. Let A be the event it's an ace, B the event it's a spade. Then $P(A \mid B) = \frac{P(AB)}{P(B)} = \frac{1/52}{4/52} = \frac{1}{4} = P(A)$

so they're independent.

Remark if A and B are independent, then

$$P(AB) = P(A)P(B).$$

Also, A and B independent implies A and B^c, A^c and B, and A^c and B^c are all independent pairs.

Def A collection of events is called independent if for every finite subcollection $A_1, \dots, A_k$,

$$P(A_1 \cdots A_k) = P(A_1) \cdots P(A_k).$$

Def A random variable $X : \Omega \to \mathbb{R}$ is a function that assigns numerical values to the outcomes of a random experiment. If a random variable has a finite or countably infinite range, it's called a discrete random variable.

Example Consider rolling 2 dice. Let X be the sum of the two rolls, let Y be the maximum of the 2 rolls.

① Find the ranges of X and Y

② Find $P(X=4)$, $P(X \leq 10)$, $P(Y \geq 5)$

① range of X is $\{2, 3, \ldots, 12\}$

range of Y is $\{1, \ldots, 6\}$

② $P(X=4) = P(\{(1,3), (3,1), (2,2)\}) = \frac{3}{36} = \frac{1}{12}$

$P(X \leq 10) = 1 - P(X > 10)$

$= 1 - (P(X=11) + P(X=12)) = 1 - \frac{2}{36} - \frac{1}{36} = \frac{11}{12}$

$P(Y \geq 5) = P(Y=5) + P(Y=6)$

$= \frac{9}{36} + \frac{11}{36} = \frac{20}{36} = \frac{5}{9}$

Def Let $S \subseteq \mathbb{R}$ be a finite set. We say that a random variable X is *uniformly distributed on S* (denoted $X \sim Unif(S)$) if the range of X is S and $P(X=x) = \frac{1}{|S|}$ for all $x \in S$.

Example Pick an integer "at random" from 1 to 50. Find the probability ① it's 13

② it's between 10 and 20 inclusive

③ it's prime

Let $X \sim Unif(\{1, 2, \ldots, 50\})$ be your number.

Then ① $P(X=13) = \frac{1}{50}$

② $P(10 \leq X \leq 20) = P(X=10) + \cdots + P(X=20) = \frac{11}{50}$

③ $P(X \text{ is prime}) = \frac{15}{50}$.

Problem 1. A gambler's dispute in 1654 is to have led to the creation of the European school of mathematical probability. Two French mathematicians, Pascal and Fermat, considered the probability that, in 24 throws of a pair of dice, at least one "double six" occurs. It was commonly believed by gamblers at the time that betting on double sixes in 24 throws of a pair of dice would be a profitable bet (ie. greater than 50% probability), but Pascal and Fermat showed otherwise. Find the probability.

Let A be the event of getting at least one double six in 24 throws. Notice the probability of getting double six on one throw is $\frac{1}{36}$.

Then $P(A) = 1 - P(A^c)$

$= 1 - \left(\frac{35}{36}\right)^{24}$

$= 0.4914039$

Problem 2. A manufacturing process produces electronic components that are sometimes defective. There is a one-in-a-thousand chance that an individual component is defective, and whether or not a component is defective is independent of any other component's status. Find the probability that among 500 components, at least one is defective.

Let X be the number of defective components among the 500. Let $X_k = \begin{cases} 1 & \text{if component } k \text{ defective} \\ 0 & \text{if component } k \text{ not def.} \end{cases}$

for $k = 1, 2, \ldots, 500$.

Then $P(X \geq 1) = 1 - P(X = 0)$

$= 1 - P(X_1 = 0, X_2 = 0, \ldots, X_{500} = 0)$

$= 1 - P(X_1 = 0)P(X_2 = 0) \cdots P(X_{500} = 0)$

$= 1 - \left(\frac{999}{1000}\right)^{500}$

Problem 3. Suppose X is a random variable that takes values on all positive integers. That is, its range is all positive integers. Let $A = \{2 \leq X \leq 4\}$ and $B = \{X \geq 4\}$. Describe the events $A^c, B^c, AB, A \cup B$.

$$A^c = \{X = 1\} \cup \{5 \leq X < \infty\}$$

$$B^c = \{1 \leq X \leq 3\} = \{X = 1\} \cup \{X = 2\} \cup \{X = 3\}$$

$$AB = \{X = 4\}$$

$$A \cup B = \{X \geq 2\}$$

Problem 4. In Julia's garden, there is a 3% chance that a tomato will be bad, with each tomato independent from the others. Julia harvests 100 tomatoes. Let X be the number of bad tomatoes harvested.

a. Find the range of X.

b. Express the event of getting no bad tomatoes in terms of X and find its probability.

c. Express the event of getting at most five bad tomatoes in terms of X and find its probability.

(a) $\{0, 1, 2, \ldots, 100\}$

(b) $P(X = 0) = (0.97)^{100}$

(c) $P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$

$$= (0.97)^{100} + 100(0.97)^{99}(0.03) + \binom{100}{2}(0.97)^{98}(0.03)^2$$