Math 342 —Conditional expectation

Problem 1. Consider the random variables X and Y with joint density

$$f(x,y) = \begin{cases} e^{-y} & 0 < x < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Find the conditional expectation $E[X \mid Y = y]$ through the following steps.

a. Find the marginal density $f_Y(y)$ of Y.

b. Find the conditional density $f_{X|Y}(x \mid y)$, making sure to take note of the interval of values of x for which the conditional density is non-zero. This interval will depend on y.

c. Compute $E[X \mid Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x \mid y)\, dx$. *Note it's possible to avoid doing any computation in this step by thinking intuitively about your answer to the previous part.*

Problem 2. Repeat similar steps as the previous problem to compute $E[Y \mid X = x]$.

Problem 3. Suppose $X \sim \text{Unif}(\{1/2, 1/3, 1/4\}$ represents the unknown heads probability of a coin. Let Y be the number of heads that result in tossing this coin 5 times.

a. Give the conditional probability mass function of Y given $X = x$.

b. Find $E[Y \mid X = 1/2], E[Y \mid X = 1/3], E[Y \mid X = 1/4]$. *Note that this can be done with a minimal amount of computation.*

c. State a general formula for $E[Y \mid X = x]$.

d. Make a conjecture for how to compute $E[Y]$ and give its value based on your conjecture.

Problem 4. Suppose Alice picks a random number X uniformly distributed in the interval $(0, 10)$. Then if Alice's number is $X = x$, Bob picks a number Y uniformly distributed in the interval $(0, x)$.

a. Give the conditional density of Y given $X = x$.

b. Find a general formula for $E[Y \mid X = x]$. *Note that this can be done with a minimal amount of computation.*

c. Make a conjecture for how to compute $E[Y]$ and give its value based on your conjecture.