Math 342 — Proof of the CLT

Let $X_1, X_2, \ldots$ be an i.i.d. sequence of random variables with

$$\mu = E[X_i], \quad \sigma^2 = V(X_i).$$

For each $n \geq 1$, let

$$S_n = X_1 + \cdots + X_n$$

and let $m_n(t)$ be moment generating function of

$$\frac{S_n/n - \mu}{\sigma/\sqrt{n}}.$$

Our goal is to prove that $\lim_{n\to\infty} m_n(t) = m_Z(t) = e^{t^2/2}$. We will break this into two cases. In Problems 1 to 5, we assume $\mu = 0$ and $\sigma^2 = 1$. In Problems 6 and 7, we consider generic values of $\mu \in \mathbb{R}$ and $\sigma^2 \in (0, \infty)$.

Problem 1. Simplify the expression $\frac{S_n/n-\mu}{\sigma/\sqrt{n}}$ under the assumption that $\mu = 0, \sigma^2 = 1$.

Problem 2. For each $i \geq 1$, let $m_{X_i}(t) = E[e^{tX_i}]$. State the values of $m_{X_i}(0), m'_{X_i}(0), m''_{X_i}(0)$.

Problem 3. Use independence to explain why

$$m_n(t) = m_{X_i}\left(\frac{t}{\sqrt{n}}\right)^n.$$

Problem 4. If you try to compute $\lim_{n\to\infty} m_n(t)$ using the expression in the previous problem, you get an indeterminate form. State the indeterminate form.

Problem 5. Our goal now is to show that

$$\lim_{n\to\infty} m_n(t) = e^{t^2/2}.$$

We will start by doing an algebraic manipulation, noting that $m_n(t) = e^{\ln m_n(t)}$, and instead we will compute $\lim_{n\to\infty} \ln m_n(t)$. The first steps are shown below. Complete the rest of the calculation and state a conclusion summarizing why the Central Limit Theorem has now been proved in the special case when $\mu = 0$ and $\sigma^2 = 1$.

$$\begin{aligned}\lim_{n\to\infty} \ln m_n(t) &= \lim_{n\to\infty} \ln m_{X_i}\left(\frac{t}{\sqrt{n}}\right)^n \\ &= \lim_{n\to\infty} n \ln m_{X_i}\left(\frac{t}{\sqrt{n}}\right) = \ldots\end{aligned}$$

Problem 6. Now suppose $\mu \in \mathbb{R}$ and $\sigma^2 \in (0, \infty)$ are not necessarily 0 and 1. Define

$$X_i^* = \frac{X_i - \mu}{\sigma} \quad \text{and} \quad S_n^* = X_1^* + \cdots + X_n^*,$$

and let $m_n^*(t)$ be the moment generating function of $S_n^*/\sqrt{n}$.

a. Find the mean and variance of X_i^*.

b. Find $\lim_{n\to\infty} m_n^*(t)$.

Problem 7. Prove that

$$\frac{S_n^*}{\sqrt{n}} = \frac{S_n/n - \mu}{\sigma/\sqrt{n}}$$

and use this fact to find $\lim_{n\to\infty} m_n(t)$. State a conclusion summarizing why the Central Limit Theorem has now been proved.