Math 102, Fall 2022 — Homework 1

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Due September 16 at 5:00 pm

Instructions. This problem set has material mostly from Week 1 of class, but perhaps a little bit from Monday of Week 2.

Problem 1. Each infinite series below converges. Find (1) the sum of its first 100 terms, and (2) the value it converges to.

1. $(2/5)^2 + (2/5)^3 + (2/5)^4 + \cdots$
2. $7 + 3 - \frac{3}{2} + \frac{3}{2^2} - \frac{3}{2^3} + \cdots$
3. $\sum_{n=4}^{\infty} \frac{1}{6^n}$

Problem 2. Each day, 8 tons of pollutants are dumped into a bay. Of this, 25% is removed by natural processes each day. Let Q_n denote the quantity of pollutant immediately after a dump on day n.

1. How much pollutant is in the bay immediately after the dump on day 20?
2. What happens to the quantity of pollutants in the bay over time? Find $\lim_{n\to\infty} Q_n$.

Problem 3. Bill deposits \$200 at the start of each month to his investment account, starting now. Suppose the investment yields 1% per month, compounded monthly. This means that at the end of month 1, Bill's account is worth 200(1.01) dollars. Give decimal answers to the questions below.

1. How much is his account worth at the end of month 2?
2. At the end of month 3?
3. At the end of month 24?

Problem 4. For each infinite series $\sum_{k=1}^{\infty} a_k$ below, we let $s_n = a_1 + a_2 + \ldots + a_n$ be its nth partial sum. Find $s_1, s_2, s_{3,}, s_4$ and a general formula for s_n, and then state whether the infinite series converges by discussing $\lim_{n\to\infty} s_n$.

1. $a_k = 1/5$
2. $a_k = (-1)^k$ (No need to give a general formula for s_n, just describe the general pattern)

Problem 5. The **Sierpinski carpet** is a geometric object called a fractal. It is constructed by starting with a square with side lengths 1 and proceeding with the following steps. In the first step, we divide the square into 9 equal sub-squares and remove the center sub-square. In the second step, we do the same procedure with each of the 8 remaining sub-squares: we remove the centers of each. The third step proceeds similar, and this repeats *forever*. The area of the original square that remains after these removal steps is what is called the Sierpinski carpet. Believe it or not, the Sierpinski carpet contains infinitely many points! See the figure below for the first 3 steps in the construction.

1. Find the area of the white (removed) region in the first step.
2. Find the area of the white (removed) region in the second step.
3. Find the area of the white (removed) region in the third step.
4. The area that gets removed in total is an infinite geometric series. What are a and r? How much area gets removed in total?

Figure 1: The first three steps of the construction of the Sierpinski carpet.