Math 102, Fall 2021 — Homework 1

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Due September 10 at 5:00 pm

Instructions. Your first homework, like most others this semester, has two parts. One part is on Webwork, and the other part is some problems that you will write solutions to by hand and submit on Gradescope.

Webwork

See Webwork for a set of problems from Sections 9.1 and 9.2.

Written problems

Write up solutions to the following problems, making sure to show your work, write neatly, scan clearly, and generally follow the guidelines for writing good homework solutions. You should submit solutions on Gradescope.

Problem 1. Determine whether each of the following sequences converges. If it does, find its limit. If it doesn't, explain how it diverges (ie. grows to $+\infty$, decreases to $-\infty$, or oscillates between various values).

1. $(-5/4)^n$
2. $3 + e^{-2n}$
3. $\frac{2n^2+(-1)^n 7}{3n^2+(-1)^n 5}$

Problem 2. For each of the following statements, decide whether it's true or false and give an explanation for your answer.

1. If a sequence s_n is convergent, then terms s_n tend to zero as n increases.
2. If all the terms of a sequence are less than 1 million, then the sequence is bounded.
3. If a monotone sequence of positive terms does not converge, then it has a term greater than a million.

Problem 3. Determine whether each of the following series converge. If it is convergent, find its sum. Otherwise, state why it diverges.

1. $\sum_{n=2}^{\infty}(1/4)^n$
2. $1 - 1/2 + 1/4 - 1/8 + 1/16 - 1/32 + \cdots$
3. $\sum_{n=0}^{\infty} \frac{1+2^n}{3^n}$

Problem 4. Each day, 8 tons of pollutants are dumped into a bay. Of this, 25% is removed by natural processes each day. What happens to the quantity of pollutants in the bay over time? Give the long-run quantity right after a dump.

Problem 5. Bill invests \$200 at the start of each month for 24 months, starting now. If the investment yields 0.5% per month, compounded monthly, what is its value at the end of 24 months?

Problem 6. The **Cantor set**, named after the German mathematician Georg Cantor (1845-1918), is constructed as follows. We start with the closed interval $[0, 1]$ (this is all the real numbers between 0 and 1, including 0 and 1) and in the first step of the construction, we remove the open interval $(1/3, 2/3)$ (this is all the real numbers between $1/3$ and $2/3$, but not including $1/3$ and $2/3$). That leaves the two intervals $[0, 1/3]$ and $[2/3, 1]$. In the second step, we remove the open middle third of each of these two intervals; so we remove $(1/9, 2/9)$ and $(7/9, 8/9)$. Four intervals remain and in the third step of the process, we again we remove the open middle third of each of them. We continue this procedure indefinitely, at each step removing the open middle third of every interval that remains from the preceding step. The Cantor set consists of the numbers that remain in $[0, 1]$ after all those intervals have been removed.

1. Draw a picture of a number line showing what's left after the first step.
2. How much is removed in the second step of the removal process? Again, draw a picture of a number line depicting what's left after the second step.
3. How much is removed in the third step of the removal process? Draw a picture of what's left after the third step.
4. There are indeed some numbers that are never removed from $[0, 1]$ as this process continues indefinitely. Give 3 example numbers that are never removed.
5. Show that the total length of the intervals removed in this indefinite process is 1. Isn't that weird?! You started with $[0, 1]$, which has total length 1. Then you removed a set of numbers of length 1, but there are still some numbers left over that never got removed!