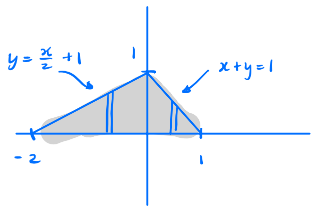
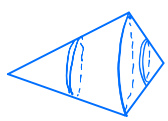
Example Find the volume of the solid formed by revolving the region bounded by $x+y=1$, $y=\frac{x}{2}+1$, $y=0$ around the x-axis.



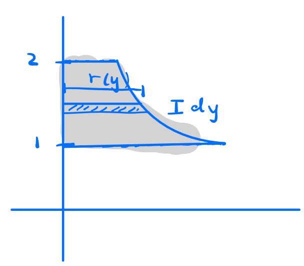
Between $x=-2$ and $x=0$, the radius of a disk is $r=\frac{x}{2}+1$

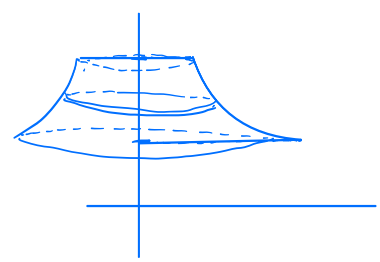
But between $x=0$ and $x=1$, it's $r=1-x$



$$\text{total volume} = \int_{-2}^{0} \pi \left(\frac{x}{2}+1\right)^2 dx + \int_0^1 \pi (1-x)^2 dx$$

Example Consider the region bounded by $y=\frac{1}{\sqrt{x}}$, $y=1$, $y=2$, $x=0$, revolved around the y-axis. Find the volume.

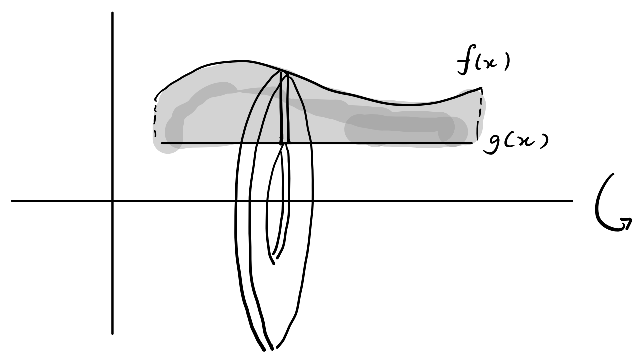




Now the radius of each disk slice is a function of y: $r(y)=\frac{1}{y^2}$

$$\text{Total volume} = \int_1^2 \pi \left(\frac{1}{y^2}\right)^2 dy$$

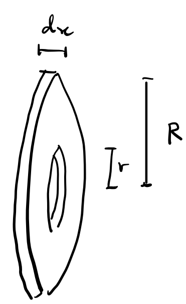
Sometimes we revolve a region around an axis, but there is a gap between the region and the axis:



In this case our solid has a hole through the middle and each slice is a disk with a hole in the middle. We refer to these slices as <u>washers</u>

volume of a washer

$= (\pi R^2 - \pi r^2)\,dx$

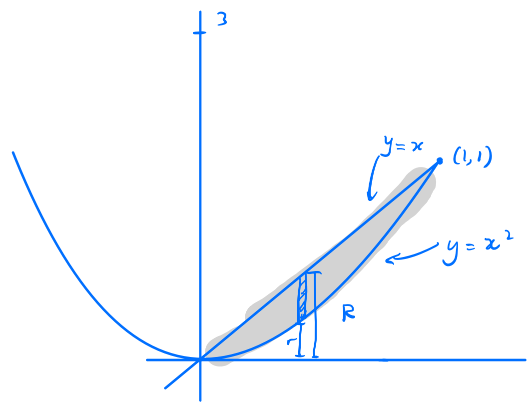


Example Revolve the region bounded by $y = x$ and $y = x^2$ around $y = 0$.

Find the volume.

$R(x) = x$

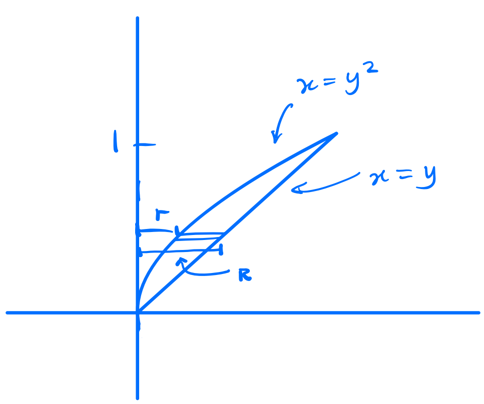
$r(x) = x^2$



$$V = \int_0^1 \pi \left(R(x)^2 - r(x)^2 \right) dx$$

$$= \int_0^1 \pi \left(x^2 - (x^2)^2 \right) dx$$

Example Revolve the region bounded by $y = \sqrt{x}$ and $y = x$ around $x = 0$ and find the volume.



$R(y) = y$, $r(y) = y^2$

$$V = \int_0^1 \pi \left(R(y)^2 - r(y)^2 \right) dy$$

$$= \int_0^1 \pi \left(y^2 - (y^2)^2 \right) dy$$

Problem 1. Each region described below is revolved around the x-axis to form a solid of revolution. Set up an integral to find its volume. No need to compute the integrals right now. Compute them after class for extra integration practice.

a. Region bounded by $y = 4 - x^2$ and $y = 0$ in the first quadrant.

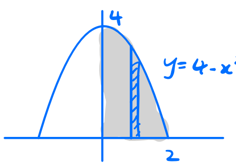
b. Region bounded by $y = 3x, x = 1$, and $x = 3$.

c. Region bounded by $y = x, y = 2x$, and $x = 1$.

d. Region bounded by $y = 2x$, and $y = x^2$.

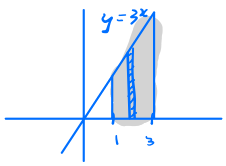
e. Region bounded by $y = x^2, x + y = 2$, and $x = 0$.

(a)



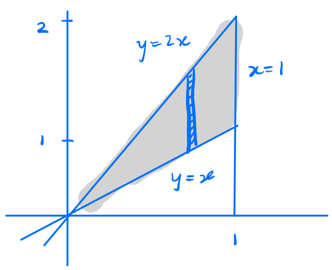
$$\int_0^2 \pi\left(4-x^2\right)^2 dx = \int_0^2 \pi\left(16-8x^2+x^4\right)dx$$

(b)



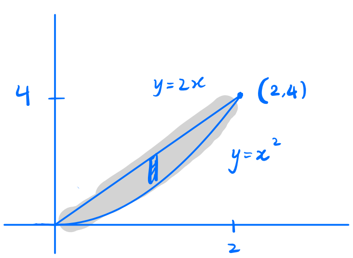
$$\int_1^3 \pi(3x)^2 dx = \int_1^3 9\pi x^2 dx$$

(c)



$$\int_0^1 \pi\left((2x)^2 - x^2\right) dx = \int_0^1 3\pi x^2 dx$$

(d)



Intersection: $x^2 = 2x$

$x^2 - 2x = 0$

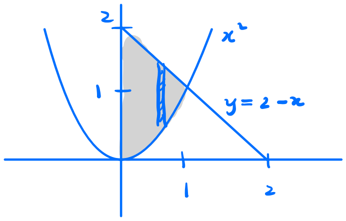
$x(x-2) = 0$

$x = 0, 2$

$$\int_0^2 \pi\left((2x)^2 - (x^2)^2\right)dx$$

$$= \int_0^2 \pi\left(4x^2 - x^4\right)dx$$

(e)



intersection point:

$x^2 = 2 - x$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

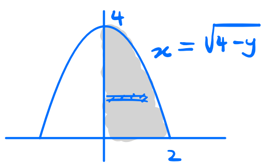
$x = -2, 1$

$$\int_0^1 \pi\left((2-x)^2 - (x^2)^2\right) dx$$

$$= \int_0^1 \pi\left(4 - 2x + x^2 - x^4\right) dx$$

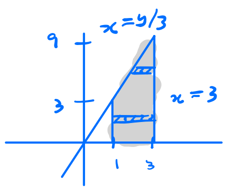
Problem 2. Repeat the problem above assuming the axis of revolution is the y-axis.

(a)



$$\int_0^4 \pi \left(\sqrt{4-y}\right)^2 dy = \int_0^4 \pi(4-y)\,dy$$

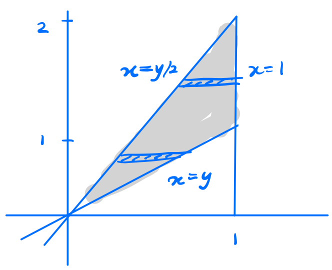
(b)



$$\int_0^3 \pi\left((3)^2 - (1)^2\right)dy + \int_3^9 \pi\left((3)^2 - (y/3)^2\right)dy$$

$$= \int_0^3 8\pi\,dy + \int_3^9 \pi\left(9 - \frac{1}{9}y^2\right)dy$$

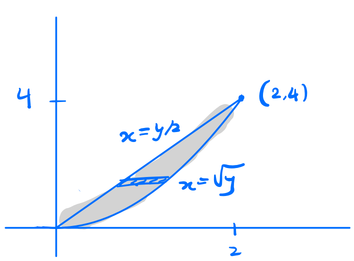
(c)



$$\int_0^1 \pi\left((y)^2 - (y/2)^2\right)dy + \int_1^2 \pi\left((1)^2 - (y/2)^2\right)dy$$

$$= \int_0^1 \frac{3\pi}{4} y^2\,dy + \int_1^2 \pi\left(1 - \frac{1}{4}y^2\right)dy$$

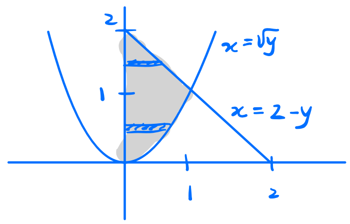
(d)



$$\int_0^4 \pi\left((\sqrt{y})^2 - (y/2)^2\right)dy$$

$$= \int_0^4 \pi\left(y - \frac{1}{4}y^2\right)dy$$

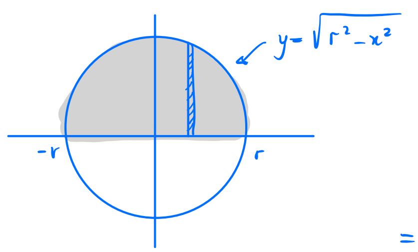
(e)



$$\int_0^1 \pi (\sqrt{y})^2\, dy + \int_1^2 \pi (2-y)^2\, dy$$

$$= \int_0^1 \pi y\, dy + \int_1^2 \pi (4-4y+y^2)\, dy$$

Problem 3. Let $r > 0$ be a given constant. Find the volume of the solid formed by revolving around the x-axis the region bounded by $y = \sqrt{r^2 - x^2}$ and $y = 0$. Make sure to compute the integral to get a formula in terms of r. What formula did you derive?



$$\int_{-r}^{r} \pi \left(\sqrt{r^2 - x^2}\right)^2 dx$$

$$= \int_{-r}^{r} \pi (r^2 - x^2)\, dx$$

$$= 2\pi \int_0^r (r^2 - x^2)\, dx$$

$$= 2\pi \left(r^2 x - \frac{1}{3} x^3 \Big|_0^r \right)$$

$$= 2\pi \left(r^3 - \frac{1}{3} r^3 \right)$$

$$= \frac{4}{3} \pi r^3$$