Problem 1. First, consider a_n below as the nth term of a sequence. State whether the sequence converges and, if so, find its limit. Second, consider a_n as the nth term of a series. State whether the series converges and, if possible, find its sum.

a. $a_n = \frac{n^3+4n^2+3}{8n^4+5n+7}$

b. $a_n = \frac{9^{n+1}}{10^n}$

c. $a_n = (-1)^n \frac{n^4+1}{3n^4+4}$

d. $a_n = n^2 2^{-n}$

a) sequence converges, $\lim_{n\to\infty} \frac{n^3+4n^2+3}{8n^4+5n+7} = 0$

series diverges, by limit comparison test, comparing to $\sum_{n=1}^{\infty} \frac{1}{n}$.

b) sequence converges, $\lim_{n\to\infty} \frac{9 \cdot 9^n}{10^n} = \lim_{n\to\infty} 9 \cdot \left(\frac{9}{10}\right)^n$

$= 9 \cdot \lim_{n\to\infty} \left(\frac{9}{10}\right)^n$

$= 0$

series converges since it's geometric with $|r| = \frac{9}{10} < 1$, its sum is $\frac{a}{1-r} = 90$.

where $a = 9$, $r = 9/10$

c) sequence diverges, $\lim_{n\to\infty} (-1)^n \frac{n^4+1}{3n^4+4}$ DNE

since values eventually oscillate between $\pm \frac{1}{3}$.

d) Series converges by ratio test since

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)^2\, 2^{-n+1}}{n^2\, 2^{-n}}$$

$$= 2^{-1} \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = 2^{-1} = \frac{1}{2} <$$

Cannot find sum though.

Sequence converges to 0 (terms of a convergent series must converge to 0 by n^{th} term test)

Problem 2. Find the sum of the infinite series

a. $-3+2-\frac{4}{3}+\frac{8}{9}-\frac{16}{27}+\cdots$

b. $\sum_{n=1}^{\infty} 6(0.9)^{n-1}$

c. $\sum_{n=1}^{\infty} \frac{(-3)^n}{4^n}$

a) $-3+2-\frac{4}{3}+\frac{8}{9}-\frac{16}{27}+\cdots$

$= \frac{a}{1-r}$ with $a=-3$, $r=\frac{-2}{3}$

$= \frac{-3}{1-(-\frac{2}{3})}$

$= \frac{-9}{5}$

b) $\sum_{n=1}^{\infty} 6(0.9)^{n-1} = 6+6(0.9)+6(0.9)^2+6(0.9)^3+\cdots$

$= \frac{6}{1-0.9} = 60$

c) $\sum_{n=1}^{\infty} \frac{(-3)^n}{4^n} = \sum_{n=1}^{\infty} \left(\frac{-3}{4}\right)^n$

$= \left(-\frac{3}{4}\right)+\left(\frac{-3}{4}\right)^2+\left(\frac{-3}{4}\right)^3+\cdots$

$= \frac{-3/4}{1-(\frac{-3}{4})}$

$= \frac{-3}{7}$

Problem 3. Maintenance on a new car starts at \$500 by the end of the first year of ownership but increases by 20% annually. Find the total maintenance cost after 15 years of ownership.

M_n = total maintenance cost after n years

$$M_1 = 500$$

$$M_2 = 500 + 500(1.2)$$

$$M_3 = 500 + 500(1.2) + 500(1.2)^2$$

$$\vdots$$

$$M_{15} = 500 + 500(1.2) + 500(1.2)^2 + \cdots + 500(1.2)^{14}$$

$$= 500\left[\frac{1-(1.2)^{15}}{1-1.2}\right]$$

Problem 4. Determine whether the series $\sum_{n=1}^{\infty} a_n$ converges where a_n is given below. State the test used and make sure to justify your use of the test with appropriate details.

a. $a_n = \frac{1}{n\sqrt{n^2+1}}$

b. $a_n = \frac{n!}{5^n}$

c. $a_n = \frac{n}{3n+1}$

d. $a_n = (-1)^n \frac{1}{5n^2+1}$

e. $a_n = (-1)^{n-1} \frac{1}{\sqrt{5n^2+1}}$

f. $a_n = \frac{3}{n+(1.2)^n}$

g. $a_n = \frac{2^n n!}{(n+2)!}$

h. $a_n = \frac{n^{0.1}-1}{n(\sqrt{n}+1)}$

a) converges by limit comparison test:

Let $a_n = \frac{1}{n\sqrt{n^2+1}}$ and $b_n = \frac{1}{n^2}$

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n\sqrt{n^2+1}}$

$= \lim_{n\to\infty} \sqrt{\frac{n^4}{n^4+n^2}}$

$= \sqrt{1} = 1 > 0$

So $\sum a_n$ converges since $\sum b_n$ converges since it's a p-series with $p = 2 > 1$.

b) diverges by ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!}{5^{n+1}} \cdot \frac{5^n}{n!} \right|$$

$$= \lim_{n\to\infty} \frac{n+1}{5} = \infty > 1$$

c) diverges by n^{th} term test:

$$\lim_{n\to\infty} \frac{n}{3n+1} = \frac{1}{3} \neq 0$$

d) converges by absolute convergence test and comparison (or limit comparison) test:

$$\sum |a_n| = \sum \frac{1}{5n^2+1} \quad \text{and}$$

$$\frac{1}{5n^2+1} \leq \frac{1}{5n^2}$$

and $\sum \frac{1}{5n^2} = \frac{1}{5}\sum \frac{1}{n^2}$ converges since it's a p-series with $p = 2 > 1$

Therefore $\sum |a_n|$ converges by comparison test, which implies $\sum a_n$ converges by absolute convergence test.

[Note: it's ok to use alt. series test here instead, but this technique lets us even say the series converges absolutely]

e) converges by alt. series test:

Let $b_n = \frac{1}{\sqrt{5n^2+1}}$

Then $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{5n^2+1}} = 0$

and $b_{n+1} = \frac{1}{\sqrt{5(n+1)^2+1}} \leq \frac{1}{\sqrt{5n^2+1}} = b_n$,

so b_n is decreasing. Then $\sum a_n$ converges by alt. series test.

[Note: this is the only test that can be used]

(f) Converges by comparison test:

Note $a_n = \dfrac{3}{n+(1.2)^n} \le \dfrac{3}{(1.2)^n} = 3\left(\dfrac{1}{1.2}\right)^n$

and $\sum 3\left(\frac{1}{1.2}\right)^n$ is a convergent geometric series, so $\sum a_n$ converges by comparison test

[Note: ratio test is difficult to use here since the limit is a little tricky algebraically]

(g) Diverges by ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(n+1)!}{(n+3)!}\,\frac{(n+2)!}{2^n n!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2(n+1)}{n+3}\right|$$

$$= 2 > 1$$

(h) Converges by comparison (or limit comparison) test:

Let $b_n = \dfrac{1}{n^{1.4}}$. Then

$$a_n = \frac{n^{0.1}-1}{n(\sqrt{n}+1)} \le \frac{n^{0.1}}{n(\sqrt{n})} = \frac{n^{0.1}}{n^{1.5}} = \frac{1}{n^{1.4}} = b_n$$

Since $\sum b_n$ converges, since it's a p-series with $p = 1.4 > 1$, $\sum a_n$ converges too.

Problem 5. Determine whether the following series converge absolutely, converge conditionally, or diverge.

a. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^5+2}$

b. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{5n+2}}$

c. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^{1/3}+3}$

a) Converges absolutely:

$\sum |a_n| = \sum \frac{n^2}{n^5+2}$, which converges

by comparison (or limit comparison) test

with $b_n = \frac{1}{n^3}$ since

$$\frac{n^2}{n^5+2} \leq \frac{n^2}{n^5} = \frac{1}{n^3}$$

and $\sum b_n$ is a convergent p-series.

b) Converges conditionally

Note $\sum |a_n| = \sum \frac{1}{\sqrt{5n+2}}$ diverges by

limit comparison test. If $b_n = \frac{1}{\sqrt{n}}$ then

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{5n+2}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{5n+2}} = \lim_{n\to\infty} \sqrt{\frac{n}{5n+2}} = \sqrt{\frac{1}{5}} > 0.$$

which implies $\sum \frac{1}{\sqrt{5n+2}}$ since $\sum b_n$ is a

divergent p-series.

However $\sum a_n$ converges by alt. series test (see problem 4e above).

c) Converges conditionally.

Note $\sum |a_n| = \sum \frac{1}{n^{1/3}+3}$ diverges

by limit comparison test. If $b_n = \frac{1}{n^{1/3}}$ then

$$\lim_{n\to\infty} \frac{\frac{1}{n^{1/3}+3}}{\frac{1}{n^{1/3}}} = \lim_{n\to\infty} \frac{n^{1/3}}{n^{1/3}+3} = 1 > 0$$

which implies $\sum \frac{1}{n^{1/3}+3}$ diverges since

$\sum b_n$ is a divergent p-series.

However $\sum a_n$ converges by alt. series

test since ① $\lim_{n\to\infty} \frac{1}{n^{1/3}+3} = 0$ and

② $\frac{1}{(n+1)^{1/3}+3} \leq \frac{1}{n^{1/3}+3}$