6.2 Integration by parts

In this section we learn a method of integration when the integrand is a product of two functions but substitution fails, eg.:

$$\int x \cos x \, dx.$$

Basic idea Let u, v be function of x.

Product rule for derivatives says:

$$(uv)' = u'v + uv'.$$

Integrating both sides with respect to x gives:

$$\int (uv)' dx = \int (u'v + uv') dx$$

Simplifying both sides:

$$uv = \int u'v \, dx + \int uv' \, dx$$

Rearranging:

$$\int uv' \, dx = uv - \int u'v \, dx$$

If we use the notation $du = u' dx$, $dv = v' dx$

we get the integration by parts formula:

$$\boxed{\int u \, dv = uv - \int v \, du}$$

Main challenge: deciding which function is u, which is dv

Examples

① $$\int \underbrace{x}_{u} \underbrace{\cos x \, dx}_{dv}$$

$u = x \qquad dv = \cos x \, dx$

$du = dx \qquad v = \sin x$

$$= \underbrace{x \sin x}_{uv} - \int \underbrace{\sin x \, dx}_{v\,du}$$

$$= x \sin x + \cos x + C$$

Note we choose u to be the function that got "simpler" when taking a derivative

② $$\int x e^x \, dx$$

$u = x \qquad dv = e^x \, dx$

$du = dx \qquad v = e^x$

$$= x e^x - \int e^x dx$$

$$= x e^x - e^x + C$$

(3) $\int x^2 \cos x \, dx$

$u = x^2 \qquad dv = \cos x \, dx$

$du = 2x \, dx \qquad v = \sin x$

$= x^2 \sin x - \int 2x \sin x \, dx \qquad u = -2x \qquad dv = \sin x \, dx$

$= x^2 \sin x + \int (-2x) \sin x \, dx \qquad du = -2 \, dx \qquad v = -\cos x$

$= x^2 \sin x + 2x \cos x - \int 2 \cos x \, dx$

$= x^2 \sin x + 2x \cos x - 2 \sin x + C$

Note: Sometimes we have to apply integration by parts more than once in a problem

(4) $\int \ln x \, dx$

$u = \ln x \qquad dv = 1 \, dx$

$du = \frac{1}{x} dx \qquad v = x$

$= \int 1 \cdot \ln x \, dx$

$= x \ln x - \int dx$

$= x \ln x - x + C$

(5) $\int \cos^2 x \, dx$ $\qquad u = \cos x \qquad dv = \cos x \, dx$

$\qquad du = -\sin x \, dx \qquad v = \sin x$

$= \sin x \cos x + \int \sin^2 x \, dx$

$= \sin x \cos x + \int (1 - \cos^2 x) \, dx$

$= \sin x \cos x + x - \int \cos^2 x \, dx$. Rearranging gives

$2\int \cos^2 x \, dx = \sin x \cos x + x$, so $\int \cos^2 x \, dx = \frac{1}{2}\sin x \cos x + \frac{1}{2}x$.

Rule of thumb for choosing u:

In order, starting from highest priority choose u to be a

L ogarithmic function

I nverse trig function

A lgebraic function (polynomial

T rig function

E xponential function

Some authors suggest LIPET instead.

polynomials

Problem. Find the following definite integrals using integration by parts. Remember the LIATE mnemonic.

a. $\int x \sin x \, dx$

b. $\int x e^{5x} \, dx$

c. $\int x^5 \ln x \, dx$

d. $\int x^2 e^x \, dx$

e. $\int 2x^3 e^{x^2} \, dx$ *Hint: try doing a substitution first. Use the letter y instead of u when doing your substitution.*

(a) $u = x \quad dv = \sin x \, dx$

$du = dx \quad v = -\cos x$

$= -x\cos x + \int \cos x \, dx$

$= -x \cos x + \sin x + C$

(b) $u = x \quad dv = e^{5x} dx$

$du = dx \quad v = \frac{1}{5} e^{5x}$

$= \frac{1}{5} x e^{5x} - \int \frac{1}{5} e^{5x} dx$

$= \frac{1}{5} x e^{5x} - \frac{1}{25} e^{5x} + C$

(c) $u = \ln x \quad dv = x^5 dx$

$du = \frac{1}{x} \quad v = \frac{1}{6} x^6$

$= \frac{1}{6} x^6 \ln x - \int \frac{1}{6} x^5 dx$

$= \frac{1}{6} x^6 \ln x - \frac{1}{36} x^6 + C$

(d) $u = x^2 \quad dv = e^x dx$

$du = 2x dx \quad v = e^x$

$= x^2 e^x - \int 2x e^x dx$

$u = -2x \quad dv = e^x dx$

$du = -2dx \quad v = e^x$

$= x^2 e^x - 2xe^x + \int 2e^x dx$

$= x^2 e^x - 2xe^x + 2e^x + C$

(e) $y = x^2$

$dy = 2x dx$

$= \int 2x \cdot x^2 \cdot e^{x^2} dx$

$= \int y e^y dy \qquad u = y \quad dv = e^y dy$

$\qquad\qquad\qquad du = dy \quad v = e^y$

$= ye^y - \int e^y dy$

$= ye^y - e^y + C$

$= x^2 e^{x^2} - e^{x^2} + C$

Example (Definite integrals)

① $\displaystyle\int_1^2 x^2 \ln x \, dx$

$u = \ln x \qquad dv = x^2 dx$

$du = \frac{1}{x} dx \qquad v = \frac{1}{3} x^3$

$$= \frac{1}{3} x^3 \ln x \Big|_1^2 - \int_1^2 \frac{1}{3} x^2 dx$$

$$= \frac{8}{3} \ln 2 - \frac{1}{3} \ln 1 - \frac{1}{9} x^3 \Big|_1^2$$

$$= \frac{8}{3} \ln 2 - \frac{1}{9}\left(2^3 - 1^3\right) = \frac{8}{3} \ln 2 - \frac{8}{9}$$

② $\displaystyle\int_0^1 \arctan x \, dx$

$u = \arctan x \qquad dv = 1 dx$

$du = \frac{1}{1+x^2} dx \qquad v = x$

$$= x \arctan x \Big|_0^1 - \int_0^1 \frac{x}{1+x^2} dx$$

$u = 1 + x^2$

$du = 2x dx$

$\frac{1}{2} du = x dx$

$$= \arctan(1) - \frac{1}{2} \int_1^2 \frac{1}{u} du$$

$$= \frac{\pi}{4} - \frac{1}{2} \ln|u| \Big|_1^2$$

$$= \frac{\pi}{4} - \frac{1}{2} \ln 2$$

③ $\displaystyle\int_0^1 \arcsin x \, dx$

$u = \arcsin x \qquad dv = 1 dx$

$du = \frac{1}{\sqrt{1-x^2}} dx \qquad v = x dx$

$$= \frac{1}{2} x^2 \arcsin x \Big|_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2}\left(\arcsin(1) - \arcsin(0)\right) + \frac{1}{2} \int_1^0 \frac{1}{\sqrt{u}} du$$

$u = 1 - x^2$

$du = -2x dx$

$-\frac{1}{2} du = x dx$

$$= \frac{1}{2}\left(\frac{\pi}{2} - 0\right) - \frac{1}{2} \int_0^1 u^{-1/2} du$$

$$= \frac{\pi}{4} - u^{1/2} \Big|_0^1 = \frac{\pi}{4} - 1$$