6.5 Partial Fraction Decomposition

Consider the integral $\int \frac{1}{x^2-1}\,dx$.

It becomes much easier to work with if we decompose the fraction into pieces:

$$\frac{1}{x^2-1} = \frac{1/2}{x-1} + \frac{-1/2}{x+1}$$

(kind of like the reverse of finding common denominator).

$$\int \frac{1}{x^2-1}\,dx = \int \frac{1/2}{x-1}\,dx + \int \frac{-1/2}{x+1}\,dx$$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$

Today we'll learn <u>how</u> to decompose a a rational function into simpler pieces.

__Example__ Decompose $\frac{1}{x^2-7x+10}$ and then integrate.

$$\frac{1}{x^2-7x+10} = \frac{1}{(x-5)(x-2)} = \frac{A}{x-5} + \frac{B}{x-2}$$

Now we need to find A and B.

Multiply both sides by $(x-5)(x-2)$:

$$(x-5)(x-2)\left(\frac{1}{(x-5)(x-2)}\right) = \left(\frac{A}{x-5} + \frac{B}{x-2}\right)(x-5)(x-2)$$

$$\Rightarrow \qquad 1 = A(x-2) + B(x-5)$$

Now solve for A and B by plugging in

values for x: $\quad \underline{x=2} \quad 1 = A(2-2) + B(2-5)$

$1 = -3B, \quad B = -1/3$

$\underline{x=5} \quad 1 = A(5-2) + B(5-5)$

$1 = 3A, \quad A = 1/3$

$$\int \frac{1}{x^2-7x+10}\,dx = \int \frac{1/3}{x-5}\,dx + \int \frac{-1/3}{x-2}\,dx$$

$$= \frac{1}{3}\ln|x-5| - \frac{1}{3}\ln|x-2| + C.$$

Example Compute $\displaystyle\int \frac{10x - 2x^2}{(x-1)^2 (x+3)}\, dx$

$$\frac{10x - 2x^2}{(x-1)^3 (x+3)} = \frac{A}{x+3} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$$

(for the algebra to work out we need these factors when there's a linear term raised to a power)

Multiply both sides by $(x-1)^3 (x+3)$

$$10x - 2x^2 = A(x-1)^3 + B(x-1)^2(x+3) + C(x-1)(x+3) + D(x+3)$$

Plug in values to find A, B, C, D:

$\underline{x=1}$ $\quad 10(1) - 2(1)^2 = D(1+3)$

$8 = 4D, \quad D = 2$

$\underline{x=-3}$ $\quad -30 - 2(-3)^2 = A(-3-1)^3$

$-48 = -64A, \quad A = \frac{3}{4}$

$\underline{x=0}$ $\quad 0 = -A + 3B - 3C + 3D$

$= -\frac{3}{4} + 3B - 3C + 6$

$= 3B - 3C + \frac{21}{4}, \quad B = C - \frac{7}{4}$

$\underline{x=2}$ $\quad 12 = A + 5B + 5C + 5D$

$= \frac{3}{4} + 5B + 5C + 10$

$\frac{5}{4} = 5B + 5C, \quad B + C = \frac{1}{4}, \quad C - \frac{7}{4} + C = \frac{1}{4}, \quad C = 1$

$A = 3/4, \quad B = -3/4, \quad C = 1, \quad D = 2$

$$\int \frac{3/4}{x+3}\,dx + \int \frac{-3/4}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx + \int \frac{2}{(x-1)^3}\,dx$$

$$= \frac{3}{4}\ln|x+3| - \frac{3}{4}\ln|x-1| - (x-1)^{-1} - (x-1)^{-2} + C$$

Example $\displaystyle\int \frac{1}{(x-1)(x^2+3)}\,dx$

$$\frac{1}{(x-1)(x^2+3)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+3}$$

(since x^2+3 cannot be factored, we use a different numerator for its term in the decomposition)

Multiply both sides by $(x-1)(x^2+3)$

$$1 = A(x^2+3) + (Bx+C)(x-1)$$

Solve for A, B, C by plugging in values for x

$\underline{x=1}$ $\quad 1 = 4A\,, \quad A = \frac{1}{4}$

$\underline{x=0}$ $\quad 1 = 3A - C\,, \quad C = -\frac{1}{4}$

$\underline{x=2}$

$$\begin{aligned} 1 &= 7A + 2B + C \\ &= 7/4 + 2B - 1/4 \end{aligned}$$

$$2B = -1/2\,, \quad B = -1/4$$

$$\int \frac{1/4}{x-1}\,dx + \int \frac{-1/4\,x - 1/4}{x^2+3}\,dx$$

$$= \int \frac{1/4}{x-1}\,dx - \frac{1}{4}\int \frac{x}{x^2+3}\,dx - \frac{1}{4}\boxed{\int \frac{1}{x^2+3}\,dx}$$

We'll discuss this next time, time permitting

$u = x^2+3$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{8}\int \frac{1}{u}\,du - \frac{1}{4\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + C$$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{8}\ln|x^2+3| - \frac{1}{4\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + C$$

Rules for partial fraction decomposition

If we decompose a rational function

$$\frac{p(x)}{q(x)}$$

① The leading power of q must be larger than leading power of p

② q must be factored completely

③ Factors of q of the form $(x-a)^n$ correspond to

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}$$

④ Unfactorable quadratics $f(x)$ in q (Like x^2+1) correspond to $\dfrac{Ax+B}{f(x)}$

Problem. Perform partial fraction decomposition on each of the rational functions and then find its indefinite integral.

a. $\frac{3x^2-8x+1}{(x-2)(x+1)(x-3)}$

b. $\frac{x+1}{6x+x^2}$ *Hint: factor the denominator first*

c. $\frac{1}{(x-1)^2(x-2)}$

d. $\frac{10x+2}{(x-5)(x^2+1)}$

(a)

$$\frac{3x^2-8x+1}{(x-2)(x+1)(x-3)} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{x-3}$$

$$3x^2-8x+1 = A(x+1)(x-3) + B(x-2)(x-3) + C(x-2)(x+1)$$

$\underline{x=2}$ $\quad -3 = -3A \Rightarrow A = -1$

$\underline{x=-1}$ $\quad 12 = 12B \Rightarrow B = 1$

$\underline{x=3}$ $\quad 4 = 4C \Rightarrow C = 1$

$$= \int \frac{-1}{x-2}\,dx + \int \frac{1}{x+1}\,dx + \int \frac{1}{x-3}\,dx$$

$$= -\ln|x-2| + \ln|x+1| + \ln|x-3| + C$$

(b)

$$\frac{x+1}{x(x+6)} = \frac{A}{x} + \frac{B}{x+6}$$

$$x+1 = A(x+6) + Bx$$

$\underline{x=0}$ $\quad 1 = 6A \Rightarrow A = 1/6$

$\underline{x=-6}$ $\quad -5 = -6B \Rightarrow B = 5/6$

$$= \int \frac{1/6}{x}\,dx + \int \frac{5/6}{x+6}\,dx = \frac{1}{6}\ln|x| + \frac{5}{6}\ln|x+6| + C$$

(c) $\dfrac{1}{(x-1)^2(x-2)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x-2}$

$1 = A(x-1)(x-2) + B(x-2) + C(x-1)^2$

$\underline{x=1}$ $\quad 1 = -B, \quad B = -1$

$\underline{x=2}$ $\quad 1 = C$

$\underline{x=3}$ $\quad 1 = 2A + B + 4C$

$= 2A - 1 + 4, \quad A = -1$

$\displaystyle\int \frac{-1}{x-1}\,dx + \int \frac{-1}{(x-1)^2}\,dx + \int \frac{1}{x-2}\,dx$

$= -\ln|x-1| + (x-1)^{-1} + \ln|x-2| + C$

d) $\dfrac{10x+2}{(x-5)(x^2+1)} = \dfrac{A}{x-5} + \dfrac{Bx+C}{x^2+1}$

$10x+2 = A(x^2+1) + (Bx+C)(x-5)$

$\underline{x=5}$ $\quad 52 = 26A \Rightarrow A = 2$

$\underline{x=0}$ $\quad 2 = A - 5C \Rightarrow C = 0$

$\underline{x=6}$ $\quad 62 = 37A + 6B$

$6B = 62 - 74$

$B = -2$

$= \displaystyle\int \frac{2}{x-5}\,dx + \int \frac{-2x}{x^2+1}\,dx$

$= 2\ln|x-5| + \displaystyle\int \frac{-du}{u}$ $\qquad u = x^2+1$, $du = 2x\,dx$

$= 2\ln|x-5| - \ln|x^2+1| + C.$