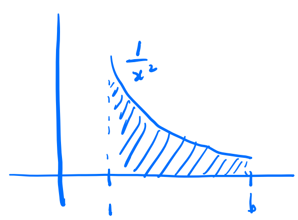
6.8 Improper integrals

An improper integral is a definite integral where either

① the interval of integration is infinite

② the integrand has a vertical asymptote somewhere along the interval of integration

Example $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$



We know how to work with definite integrals over finite intervals

We'll define $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ as

$\displaystyle\lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx$ and compute if limit exists

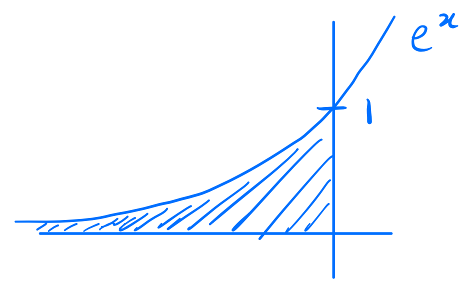
$$= \lim_{b\to\infty}\int_1^b x^{-2}\,dx$$

$$= \lim_{b\to\infty} -x^{-1}\Big|_1^b$$

$$= \lim_{b\to\infty} 1-\frac{1}{b}$$

$= 1$ (so we say the integral converges to 1)

Example $\displaystyle\int_{-\infty}^0 e^x\,dx$



$$= \lim_{b\to-\infty}\int_b^0 e^x\,dx$$

$$= \lim_{b\to-\infty} e^x\Big|_b^0$$

$$= \lim_{b\to-\infty} e^0-e^b = 1-\lim_{b\to-\infty}e^b = 1-0 = 1$$

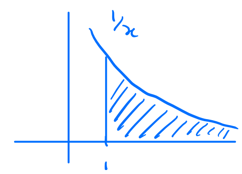
Definition $\int_a^\infty f(x)\,dx$ converges if

$\lim_{b\to\infty} \int_a^b f(x)\,dx$ is finite. If not,

we say the integral diverges.

Example Does $\int_1^\infty \frac{1}{x}\,dx$ converge?

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$$



$$= \lim_{b\to\infty} \ln|x| \Big|_1^b$$

$$= \lim_{b\to\infty} \ln b - \ln 1 = \infty$$

It does not converge. It diverges.

Example Does $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ converge?

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty} \int_1^b x^{-1/2}\,dx$$

$$= \lim_{b\to\infty} 2x^{1/2} \Big|_1^b$$

$$= \lim_{b\to\infty} 2b^{1/2} - 2 = \infty$$

It does not converge.

Question For which p does $\int_1^\infty \frac{1}{x^p}\,dx$ converge?

Theorem Consider the improper integral $\int_1^\infty \frac{1}{x^p}\,dx$, where $p > 0$. This integral converges when $p > 1$ and diverges when $p \le 1$.

Proof Suppose $p \neq 1$ (we already proved the $p = 1$ case). Then

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b x^{-p}\,dx$$

$$= \lim_{b\to\infty} \frac{1}{-p+1} x^{-p+1}\Big|_1^b$$

$$= \lim_{b\to\infty} \frac{1}{-p+1}\left(b^{-p+1} - 1\right)$$

$$= \frac{1}{-p+1} \lim_{b\to\infty} \frac{1}{b^{p-1}} - 1$$

$$\lim_{b\to\infty} \frac{1}{b^{p-1}} = \begin{cases} 0 & \text{when } p-1>0 \\ \infty & \text{when } p-1<0 \end{cases}$$

So the limit is finite when $p > 1$, is infinite when $p \le 1$.

Problem. Determine whether the following improper integrals converge or diverge.

a. $\int_1^\infty \frac{1}{(x+2)^{.4}}$

b. $\int_5^\infty \frac{1}{(x-1)^{1.5}}\,dx$

c. $\int_2^\infty \frac{\ln x}{x}\,dx$

d. $\int_0^\infty x e^{-x^2}\,dx$

(a) $$\int_1^\infty \frac{1}{(x+2)^{0.4}}\,dx = \lim_{b\to\infty} \int_1^b (x+2)^{-0.4}\,dx \qquad u = x+2,\ du = dx$$

$$= \lim_{b\to\infty} \int_3^{b+2} u^{-0.4}\,du$$

$$= \lim_{b\to\infty} \frac{1}{0.6} u^{0.6} \Big|_3^{b+2}$$

$$= \lim_{b\to\infty} \frac{1}{0.6}\left((b+2)^{0.6} - 3^{0.6}\right)$$

$$= \infty, \text{ diverges}$$

(b) $$\int_5^\infty \frac{1}{(x-1)^{1.5}}\,dx = \lim_{b\to\infty} \int_5^b (x-1)^{-1.5}\,dx \qquad u = x-1,\ du = dx$$

$$= \lim_{b\to\infty} \int_4^{b-1} u^{-1.5}\,du$$

$$= \lim_{b\to\infty} -2u^{-0.5} \Big|_4^{b-1}$$

$$= \lim_{b\to\infty} -2\left((b-1)^{-0.5} - 4^{-0.5}\right)$$

$$= 2(4)^{-0.5} = 1, \text{ converges}$$

(c) $\displaystyle\int_2^{\infty} \frac{\ln x}{x}\,dx = \lim_{b\to\infty} \int_2^{b} \frac{\ln x}{x}\,dx$ $\quad u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} u\,du$$

$$= \lim_{b\to\infty} \frac{1}{2}u^2 \Big|_{\ln 2}^{\ln b}$$

$$= \lim_{b\to\infty} \frac{1}{2}\left((\ln b)^2 - (\ln 2)^2\right)$$

$= \infty$, diverges

(d) $\displaystyle\int_0^{\infty} x e^{-x^2}\,dx = \lim_{b\to\infty} \int_0^{b} x e^{-x^2}\,dx$ $\quad u = -x^2$

$du = -2x\,dx$

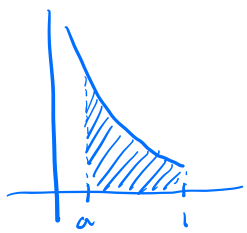
$$= \lim_{b\to\infty} -\frac{1}{2} \int_0^{-b^2} e^u\,du$$

$$= \lim_{b\to\infty} -\frac{1}{2} e^u \Big|_0^{-b^2}$$

$= \displaystyle\lim_{b\to\infty} -\frac{1}{2}\left(e^{-b^2} - 1\right) = \frac{1}{2}$, converges

<u>Example</u> Does $\int_0^1 \frac{1}{\sqrt{x}}$ converge?

How about $\int_0^1 \frac{1}{x^2}$? What's different about these? How do we handle them?



Here, the function grows unboundedly near 0. So we'll cut off the integral at a and let $a \to 0^+$

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{a\to 0^+} \int_a^1 x^{-1/2}\,dx$$

$$= \lim_{a\to 0^+} 2x^{1/2}\Big|_a^1$$

$$= 2\lim_{a\to 0^+} (1 - a^{1/2})$$

$$= 2 \qquad \text{converges}$$

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{a\to 0^+} \int_a^1 x^{-2}\,dx$$

$$= \lim_{a\to 0^+} -x^{-1}\Big|_a^1$$

$$= \lim_{a\to 0^+} (a^{-1} - 1) = \infty \quad \text{diverges.}$$