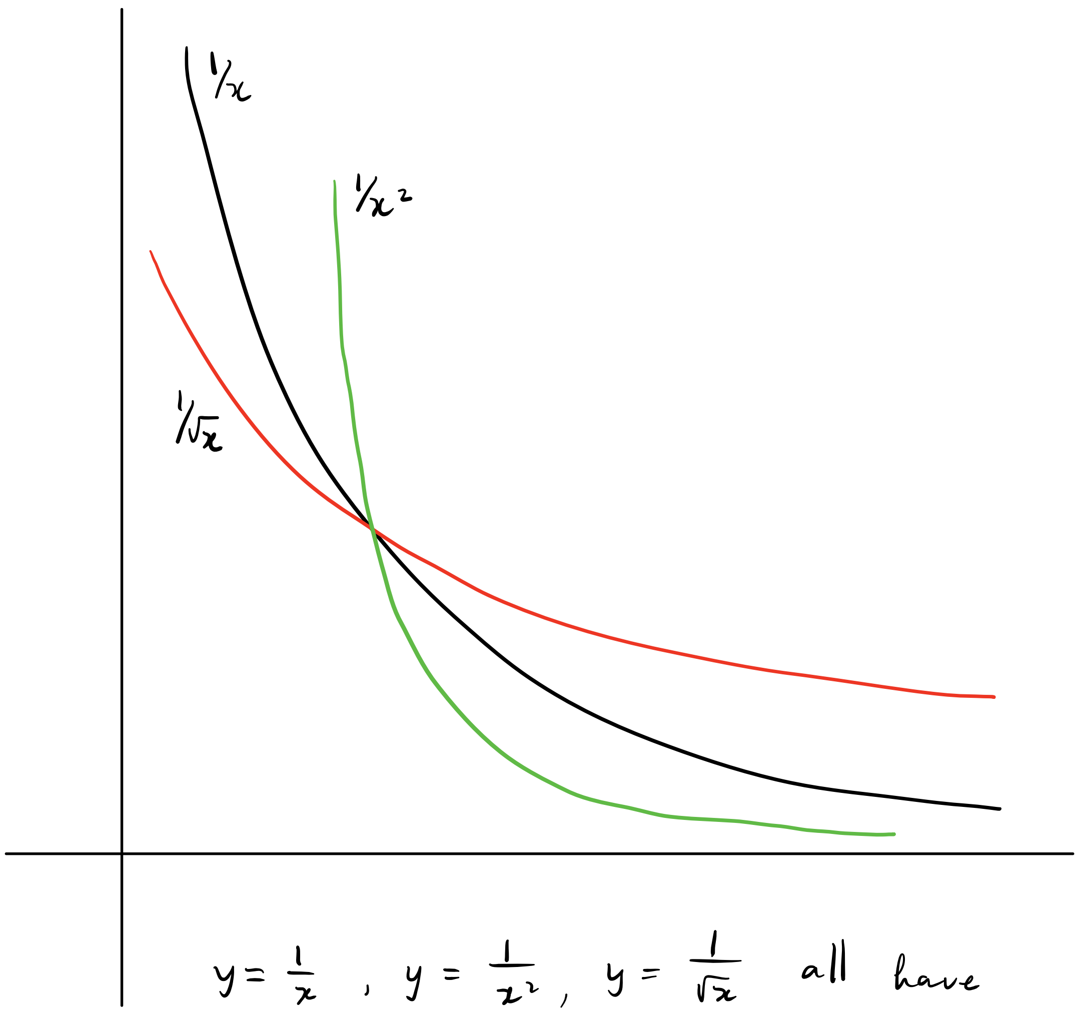
6.8 Improper integrals, continued

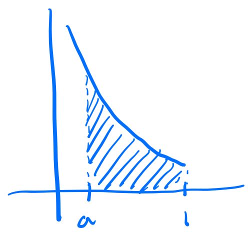
Today we'll discuss a different kind of improper integral where somewhere in the interval of integration, the function has a vertical asymptote



$y = \frac{1}{x}$, $y = \frac{1}{x^2}$, $y = \frac{1}{\sqrt{x}}$ all have vertical asymptotes at $x = 0$

Example Does $\int_0^1 \frac{1}{\sqrt{x}}$ converge?

How about $\int_0^1 \frac{1}{x}$? $\int_0^1 \frac{1}{x^2}$?



Here, the function grows unboundedly near 0. So we'll cut off the integral at a and let $a \to 0^+$

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{a\to 0^+} \int_a^1 x^{-1/2}\,dx$$

$$= \lim_{a\to 0^+} 2x^{1/2}\Big|_a^1$$

$$= 2\lim_{a\to 0^+} (1 - a^{1/2})$$

$$= 2 \quad \text{converges}$$

$$\int_0^1 \frac{1}{x}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{x}\,dx$$

$$= \lim_{a\to 0^+} \ln|x|\Big|_a^1$$

$$= \lim_{a\to 0^+} \ln 1 - \ln a$$

$$= \lim_{a\to 0^+} -\ln a = \infty, \text{ diverges}$$

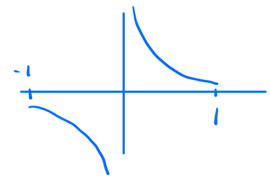
$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{a\to 0^+} \int_a^1 x^{-2}\,dx$$

$$= \lim_{a\to 0^+} -x^{-1}\Big|_a^1$$

$$= \lim_{a\to 0^+} (a^{-1} - 1) = \infty \quad \text{diverges.}$$

Example Do the following converge or diverge?

① $\displaystyle\int_{-1}^{1} \frac{1}{x^3}\, dx$



There's an asymptote in the middle!

$$= \int_{-1}^{0} \frac{1}{x^3}\, dx + \int_{0}^{1} \frac{1}{x^3}\, dx$$

$$= \lim_{b \to 0^-} \int_{-1}^{b} \frac{1}{x^3}\, dx + \lim_{a \to 0^+} \int_{a}^{1} \frac{1}{x^3}\, dx$$

$$= \lim_{b \to 0^-} -\frac{1}{2} x^{-2} \Big|_{-1}^{b} + \lim_{a \to 0^+} -\frac{1}{2} x^{-2} \Big|_{a}^{1}$$

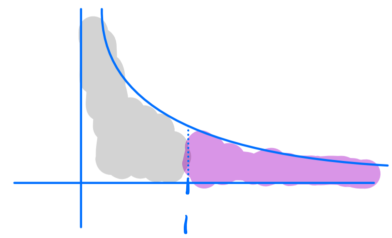
$$= \lim_{b \to 0^-} -\frac{1}{2}(b^{-2} - 1) + \lim_{a \to 0^+} -\frac{1}{2}(1 - a^{-2})$$

$$= \lim_{b \to 0^-} -\frac{1}{2}\left(\frac{1}{b^2} - 1\right) + \lim_{a \to 0^+} -\frac{1}{2}\left(1 - \frac{1}{a^2}\right)$$

$$= -\infty + \infty, \text{ diverges}$$

When <u>at least</u> one piece of the integral diverges, we <u>declare</u> the whole integral diverges.

(2) $$\int_0^\infty \frac{1}{x^4}\,dx$$



Improper at both endpoints!

$$= \int_0^1 \frac{1}{x^4}\,dx + \int_1^\infty \frac{1}{x^4}\,dx$$

$$= \lim_{a\to 0^+} \int_a^1 \frac{1}{x^4}\,dx + \lim_{b\to\infty} \int_1^b \frac{1}{x^4}\,dx$$

$$= \lim_{a\to 0^+} -\frac{1}{3}x^{-3}\Big|_a^1 + \lim_{b\to\infty} -\frac{1}{3}x^{-3}\Big|_1^b$$

$$= \lim_{a\to 0^+} -\frac{1}{3}(1-a^{-3}) + \lim_{b\to\infty} -\frac{1}{3}(b^{-3}-1)$$

$= \infty + \frac{1}{3}$, diverges (at least one piece diverged)

Problem. Determine whether the following improper integrals converge or diverge.

a. $\int_0^5 \frac{1}{x^{2/3}}\,dx$

b. $\int_1^5 \frac{1}{(x-2)^6}\,dx$

c. $\int_1^2 \frac{x^2}{(x^3-1)^{1/2}}\,dx$ *Hint: identify the asymptote and use substitution.*

d. $\int_0^\infty \frac{1}{(x-1)^2}\,dx$ *Hint: break this up into three improper integrals.*

a) $$\lim_{a\to 0^+} \int_a^5 x^{-2/3}\,dx = \lim_{a\to 0^+} 3x^{1/3}\Big|_a^5$$

$$= \lim_{a\to 0^+} 3(5^{1/3} - a^{1/3})$$

$$= 3\cdot 5^{1/3}, \text{ converges}$$

b) $$\lim_{b\to 2^-} \int_1^b (x-2)^{-6}\,dx + \lim_{a\to 2^+} \int_a^5 (x-2)^{-6}\,dx$$

$$= \lim_{b\to 2^-} -\tfrac{1}{5}(x-2)^{-5}\Big|_1^b + \lim_{a\to 2^+} -\tfrac{1}{2}(x-2)^{-5}\Big|_a^5$$

$$= \lim_{b\to 2^-} -\tfrac{1}{5}\left(\frac{1}{(b-2)^5} + 1\right) + \lim_{a\to 2^+} -\tfrac{1}{2}\left(3^{-5} - \frac{1}{(a-2)^5}\right)$$

$$= \infty + \infty, \text{ diverges}$$

c)

$$\lim_{a \to 1^+} \int_a^2 \frac{x^2}{(x^3-1)^{1/2}}\, dx \qquad u = x^3 - 1$$

$$du = 3x^2\, dx$$

$$\tfrac{1}{3}\, du = x^2\, dx$$

$$= \lim_{a \to 1^+} \frac{1}{3} \int_{a^3-1}^{7} \frac{1}{u^{1/2}}\, du$$

$$= \lim_{a \to 1^+} \frac{2}{3} u^{1/2} \Big|_{a^3-1}^{7}$$

$$= \lim_{a \to 1^+} \frac{2}{3} \left(\sqrt{7} - \sqrt{a^3-1} \right)$$

$$= \frac{2\sqrt{7}}{3}$$, converges

d)

$$\int_0^1 \frac{1}{(x-1)^2}\, dx + \int_1^2 \frac{1}{(x-1)^2}\, dx + \int_2^\infty \frac{1}{(x-1)^2}\, dx$$

$$= \lim_{b \to 1^-} \int_0^b (x-1)^{-2} dx + \lim_{a \to 1^+} \int_a^2 (x-1)^{-2} + \lim_{c \to \infty} \int_2^c (x-1)^{-2} dx$$

$$= \lim_{b \to 1^-} -(x-1)^{-1} \Big|_0^b + \lim_{a \to 1^+} -(x-1)^{-1} \Big|_a^2 + \lim_{c \to \infty} -(x-1)^{-1} \Big|_2^c$$

$$= \lim_{b \to 1^-} -\left(\frac{1}{b-1} + 1 \right) + \lim_{a \to 1^+} -\left(1 - \frac{1}{a-1} \right) + \lim_{c \to \infty} -\left(\frac{1}{c-1} - 1 \right)$$

$= \infty + \infty + 1 = \infty$, diverges because at least one diverged