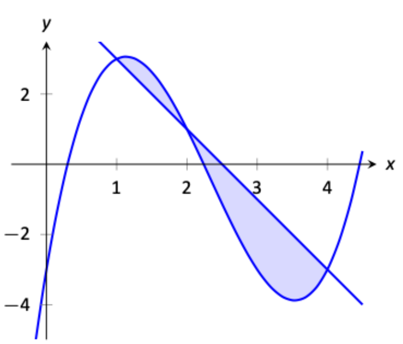
7.1 Areas Between Curves

Recall that the <u>unsigned</u> area of the region bounded between $g(x)$ and $f(x)$, where $g(x) \le f(x)$, for x between a and b is given by

$$\int_a^b \left(f(x) - g(x) \right) dx$$

top curve — bottom curve

<u>Example</u> Find the area of the region enclosed by $f(x) = -2x+5$ and $g(x) = x^3 - 7x^2 + 12x - 3$ with their graphs and intersection points shown below.



$$\int_1^2 (g(x) - f(x))\,dx + \int_2^4 (f(x) - g(x))\,dx$$

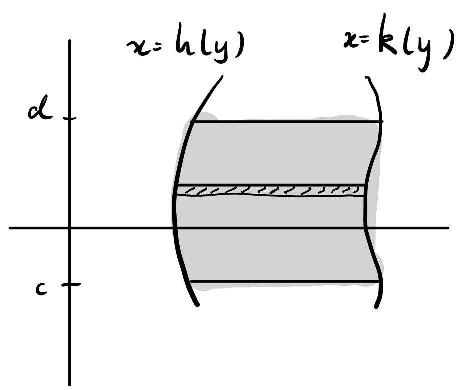
$$= \int_1^2 (x^3 - 7x^2 + 14x - 8)\,dx + \int_2^4 (-x^3 + 7x^2 - 14x + 8)\,dx$$

$$= \dots = \frac{37}{12}$$

Integrals with respect to y

Sometimes a region is specified by "left" and "right" bounding curves $x = h(y)$ and $x = k(y)$ over a range of y-values.

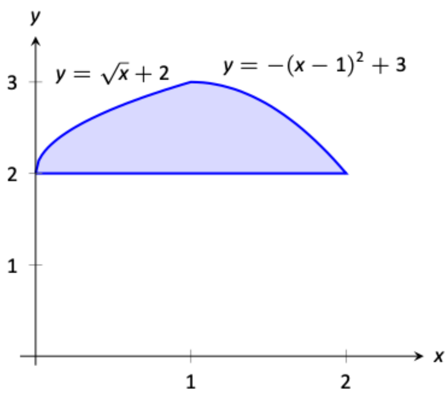
We can find unsigned areas of such regions using the same ideas:



$$\text{Area} = \int_c^d \big(\underbrace{k(y)}_{\text{right curve}} - \underbrace{h(y)}_{\text{left curve}}\big)\, dy$$

Example Set up integrals for the area of the region enclosed

by $y = \sqrt{x} + 2$, $y = -(x-1)^2 + 3$, $y = 2$ in two ways.



Note $y = \sqrt{x} + 2$ can be expressed using x in terms of y: $\sqrt{x} = y - 2$

$$x = (y-2)^2$$

And same for $y = -(x-1)^2 + 3$:

$(x-1)^2 = 3 - y$, $x = 1 + \sqrt{3-y}$,

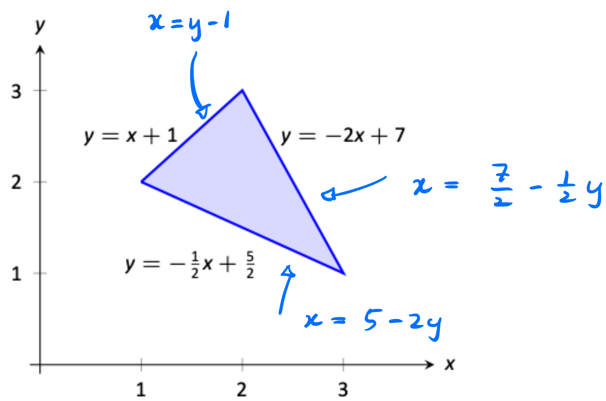
Integral with respect to x

$$\int_0^1 (\underbrace{\sqrt{x} + 2}_{\text{top curve}} - \underbrace{2}_{\text{bottom curve}})\, dx + \int_1^2 (\underbrace{-(x-1)^2 + 3}_{\text{top curve}} - \underbrace{2}_{\text{bottom curve}})\, dx$$

Integral with respect to y

$$\int_2^3 \left(\underbrace{(1 + \sqrt{3-y})}_{\text{right curve}} - \underbrace{(y-2)^2}_{\text{left curve}} \right) dy$$

Example Set up integrals for the area of the region below in two ways.



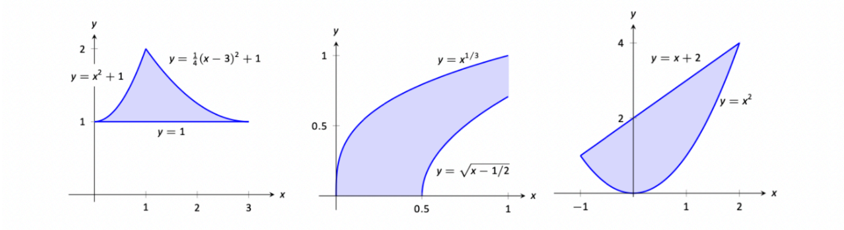
Integral with respect to x

$$\int_1^2 \left((x+1) - \left(-\tfrac{1}{2}x + \tfrac{5}{2}\right)\right) dx + \int_2^3 \left((-2x+7) - \left(-\tfrac{1}{2}x + \tfrac{5}{2}\right)\right) dx$$

Integral with respect to y

$$\int_1^2 \left(\left(\tfrac{7}{2} - \tfrac{1}{2}y\right) - (5-2y)\right) dy + \int_2^3 \left(\left(\tfrac{7}{2} - \tfrac{1}{2}y\right) - (y-1)\right) dy$$

Problem 1. For each region below, set up integrals for finding the area of the region in two ways: (1) with respect to x and (2) with respect to y. No need to compute the integrals but you can use software like Wolfram Alpha to compute both and check that you get the same answer.



(a) $\int_0^1 ((x^2+1)-1)\,dx + \int_1^3 \left(\left(\frac{1}{4}(x-3)^2+1\right)-1\right)dx$

$\int_1^2 \left((3+2\sqrt{y-1})-\sqrt{y-1}\right)dy$

(b) $\int_0^{0.5} x^{1/3}\,dx + \int_{0.5}^1 \left(x^{1/3}-\sqrt{x-1/2}\right)dx$

$\int_0^{\sqrt{1/2}} \left((y^2+\frac{1}{2})-y^3\right)dy + \int_{\sqrt{1/2}}^1 (1-y^3)\,dy$

(c) $x+2=x^2$

$x^2-x-2=0$

$(x-2)(x+1)=0$

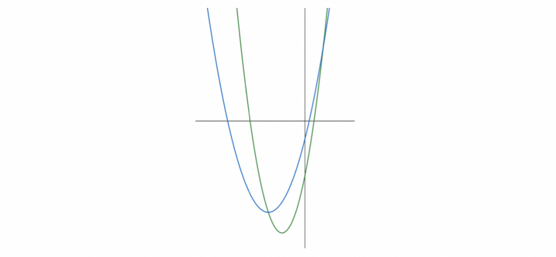
$x=-1,\ 2$

$(-1,1)$ and $(2,4)$ are intersection points

$\int_{-1}^2 ((x+2)-x^2)\,dx$

$\int_0^1 (\sqrt{y}-(-\sqrt{y}))\,dy + \int_1^4 (\sqrt{y}-(y-2))\,dy$

Problem 2. Set up an integral to find the area of region bounded by the functions $f(x) = 2x^2 + 5x - 3$ and $g(x) = x^2 + 4x - 1$. The graphs of these functions are shown below. Start by finding the two points of intersection.



$$2x^2 + 5x - 3 = x^2 + 4x - 1$$

$$x^2 + x - 2 = 0$$

$$(x+2)(x-1) = 0$$

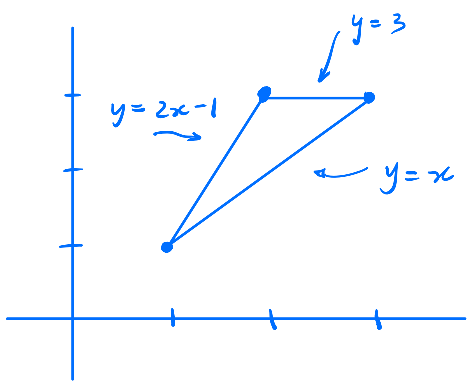
$$x = -2, 1$$

$$\int_{-2}^{1} \left((x^2+4x-1) - (2x^2+5x-3) \right) dx$$

top curve (y-intercept -1)

bottom curve (y-intercept -3)

Problem 3. Set up an integral to find the area of the triangle with vertices $(1, 1)$, $(2, 3)$, and $(3, 3)$. Start by making a sketch of the triangle on the xy-plane and find equations of the lines that define the triangle's sides.



$$\int_{1}^{3} \left(y - \tfrac{1}{2}(y+1) \right) dy$$