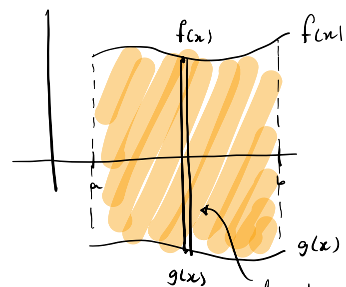
Areas between curves



height = $f(x) - g(x)$
width = dx

Area between top curve $f(x)$
and bottom curve $g(x)$
between $x=a$ and $x=b$ is

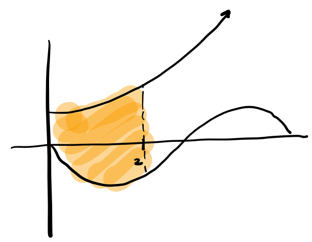
$$\int_a^b [f(x) - g(x)]\, dx.$$

Note this is an unsigned area (always positive when $f(x)$ is top curve and $g(x)$ is bottom curve)

<u>Example</u> Find the area between

$y = -\sin x$ and $y = e^x$

for x-values between 0 and 2.



$$\int_0^2 \Big[\underbrace{e^x}_{\text{top}} - \underbrace{(-\sin x)}_{\text{bottom}} \Big] \, dx$$

$$= \int_0^2 (e^x + \sin x) \, dx$$

$$= e^x - \cos x \Big|_0^2$$

$$= [e^2 - \cos(2)] - [e^0 - \cos(0)]$$

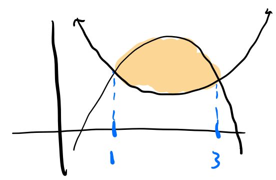
$$= e^2 - \cos 2$$

Example Find the area of the bounded region between

$$y = x^2 - 4x + 5 \quad \text{(bottom)}$$

and $$y = -x^2 + 4x - 1 \quad \text{(top)}$$

shown below



Where do these curves intersect?

$$x^2 - 4x + 5 = -x^2 + 4x - 1$$

$$2x^2 - 8x + 6 = 0$$

$$2(x^2 - 4x + 3) = 0$$

$$2(x-3)(x-1) = 0$$

$$x = 1, 3$$

$$\text{Area} = \int_1^3 \left[(-x^2+4x-1) - (x^2-4x+5)\right] dx$$

$$= \int_1^3 \left(-2x^2 + 8x - 6\right) dx$$

$$= -\frac{2}{3}x^3 + 4x^2 - 6x \Big|_1^3$$

$$= \left[-\frac{2}{3}(27) + 4(9) - 6(3)\right] - \left[-\frac{2}{3} + 4 - 6\right]$$

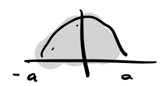
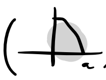
$$= \left[-18 + 36 - 18\right] - \left[-\frac{2}{3} - 2\right]$$

$$= \frac{8}{3}.$$

Using symmetry

If f is even (meaning $f(x) = f(-x)$ for all x)

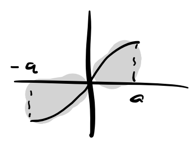
the graph of $y = f(x)$ is symmetric across the y-axis

and $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$  $= 2\Big($  $\Big)$

If f is odd (meaning $f(-x) = -f(x)$ for all x)

the graph of $y = f(x)$ is symmetric across the origin

and $\int_{-a}^{a} f(x)\,dx = 0$

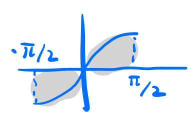


Example It is known that $\int_{0}^{\pi/2} \sin x\,dx = 1$.

Find ① $\int_{-\pi/2}^{\pi/2} \sin x\,dx$

$$= \int_{-\pi/2}^{0} \sin x\,dx + \int_{0}^{\pi/2} \sin x\,dx = -1 + 1 = 0$$

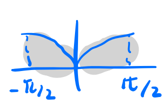
graph of $y = \sin x$



② $\int_{-\pi/2}^{\pi/2} |\sin x|\,dx$

$$= \int_{-\pi/2}^{0} |\sin x|\,dx + \int_{0}^{\pi/2} |\sin x|\,dx = 1 + 1 = 2.$$

graph of $y = |\sin x|$



Problem 1. Consider the function $f(x) = 3x^2 - 3$ whose plot is shown below.

a. Use the Fundamental Theorem of Calculus and integral properties to compute:

1. $\int_0^1 f(x)\,dx$
2. $\int_0^3 f(x)\,dx$
3. $\int_1^3 f(x)\,dx$

b. Use symmetry, the plot below, and integral properties to find:

1. $\int_{-1}^1 f(x)\,dx$
2. $\int_{-3}^3 f(x)\,dx$
3. $\int_{-1}^{-3} f(x)\,dx$
4. the *signed* area of the shaded region in the plot
5. the area of the shaded region in the plot

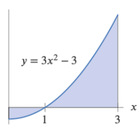


Figure 6.25

ⓐ $\int_0^1 (3x^2-3)\,dx$

$= x^3 - 3x \Big|_0^1$

$= [1^3 - 3(1)] - [0^3 - 3(0)]$

$= -2$

$\int_0^3 (3x^2-3)\,dx$

$= x^3 - 3x \Big|_0^3$

$= 3^3 - 3^2$

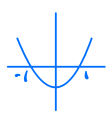
$= 18$

$\int_1^3 f(x)\,dx = \int_0^3 f(x)\,dx - \int_0^1 f(x)\,dx$

$= 18 - (-2)$

$= 20$

ⓑ f is an <u>even</u> function



$\int_{-1}^1 f(x)\,dx = 2\int_0^1 f(x)\,dx = 2(-2) = -4$

$\int_{-3}^3 f(x)\,dx = 2\int_0^3 f(x)\,dx = 2(18) = 36$

$\int_{-3}^{-1} f(x)\,dx = \int_1^3 f(x)\,dx = 20$, so $\int_{-1}^{-3} f(x)\,dx = -20$

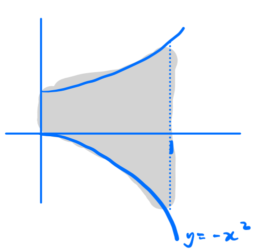
signed area $= \int_0^3 f(x)\,dx = 18$

area $= -\int_0^1 f(x)\,dx + \int_1^3 f(x)\,dx = 2 + 20 = 22$

Problem 2. Consider the region between $y = e^x$ and $y = -x^2$ for values of x between 0 and 1.

a. Sketch a picture of the region and label each curve.

b. Find the area of the region.



top curve bottom curve

area $= \int_0^1 [e^x - (-x^2)]\,dx$

$= \int_0^1 (e^x + x^2)\,dx$

$= e^x + \frac{1}{3}x^3 \Big|_0^1 = (e + \frac{1}{3}) - (e^0) = e - \frac{2}{3}$

Problem 3. Consider the region bounded the curves $y = x^2$ and $y = 8 - x^2$.

a. Where do these two curves intersect?

b. Sketch a picture of the region and label each curve.

c. Find the area of the region.

(a) $x^2 = 8 - x^2$

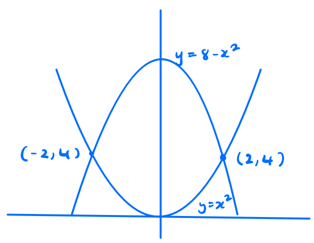
$\Rightarrow 2x^2 = 8$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$

So $(2,4)$ and $(-2,4)$ are intersection points

(b)



(c) $\text{area} = \int_{-2}^{2} [\underbrace{(8-x^2)}_{\text{top}} - \underbrace{(x^2)}_{\text{bottom}}]\,dx$

$= \int_{-2}^{2} (8 - 2x^2)\,dx$

$= 2\int_0^2 (8 - 2x^2)\,dx$

$= \int_0^2 (16 - 4x^2)\,dx$

$= 16x - \frac{4}{3}x^3 \Big|_0^2$

$= 32 - \frac{32}{3}$

$= \frac{64}{3}$

Problem 4. Find the following definite integrals.

a. $\int (4e^x - 3\sin x)\,dx$

b. $\int \left(\frac{3}{t} - \frac{2}{t^2}\right) dt$

c. $\int (x+3)^2\,dx$

d. $\int t^3(t^2+1)\,dt$

(a) $4\int e^x\,dx - 3\int \sin x\,dx = 4e^x + 3\cos x + C$

(b) $3\int \frac{1}{t}\,dt - 2\int t^{-2}\,dt = 3\ln|t| + 2t^{-1} + C$

(c) $\int (x^2 + 6x + 9)\,dx = \frac{1}{3}x^3 + 3x^2 + 9x + C$

(d) $\int (t^5 + t^3)\,dt = \frac{1}{6}t^6 + \frac{1}{4}t^4 + C$