8.2 Infinite Series

Definition An infinite series (or simply series) is the sum of an infinite list of numbers:

ending index (no end)

$$\sum_{k=1}^{\infty} a_k = \underbrace{a_1 + a_2 + a_3 + a_4 + \cdots}_{\text{infinitely many terms}}$$

index variable

starting index

The n^{th} partial sum is the finite sum of the first n terms of the series: $S_n = \underbrace{a_1 + a_2 + \cdots + a_n}_{\text{finitely many terms}}$

(e.g. $n = 40$)

Repeated Drug Dosage

A person with an ear infection is told to take antibiotic tablets regularly for several days. Since the drug is being excreted by the body between doses, how can we calculate the quantity of the drug remaining in the body at any particular time?

To be specific, let's suppose the drug is ampicillin (a common antibiotic) taken in 250 mg doses four times a day (that is, every six hours). It is known that at the end of six hours, about 4% of the drug is still in the body. What quantity of the drug is in the body right after the tenth tablet? The fortieth?

Let Q_n represent the quantity, in milligrams, of ampicillin in the blood right after the n^{th} tablet.

$$Q_1 = 250$$

$$Q_2 = 250 + 250(0.04)$$

$$Q_3 = 250 + Q_2(0.04) = 250 + 250(0.04) + 250(0.04)^2$$

$$Q_4 = 250 + Q_3(0.04) = 250 + 250(.04) + 250(.04)^2 + 250(.04)^3$$

$$Q_n = 250 + Q_{n-1}(0.04) = 250 + 250(.04) + \cdots + 250(.04)^{n-1}$$

Definition A *geometric series* is a series of the form

$$a + ar + ar^2 + ar^3 + ar^4 + \dots$$

for some given constants a and r.

(Notice r is the common ratio between any term and the previous term)

Question Is it possible to find a concrete, simple formula (in terms of a, r, n) for

$$a + ar + ar^2 + \dots + ar^{n-1} ?$$

Yes! Let $S_n = a + ar + \dots + ar^{n-1}$ and assume $r \neq 1$

We want a simple formula for S_n.

Notice $rS_n = ar + ar^2 + \dots + ar^{n-1} + ar^n$

and $\underbrace{S_n - rS_n}_{} = a - ar^n$

Therefore $S_n(1-r) = a - ar^n$

and so $S_n = \dfrac{a - ar^n}{1-r} = \dfrac{a(1-r^n)}{1-r}$

Definition An infinite series $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + \cdots$ **converges to L** if $\lim_{n\to\infty} (a_1 + a_2 + \cdots a_n) = L$.

We say $\sum_{k=1}^{\infty} a_k = L$ in this case.

If the limit $\lim_{n\to\infty} (a_1 + \cdots + a_n)$ diverges, we say the infinite series **diverges**.

Notice a shorter way of phrasing this is to say the infinite series converges when the partial sum converges, ie. $\lim_{n\to\infty} S_n$ exists.

Theorem Consider the geometric series $\sum_{k=0}^{\infty} ar^k$.

Then

(1) the infinite series converges if and only if $|r| < 1$

(2) when $|r|<1$, the infinite series converges to $\frac{a}{1-r}$.

note the starting index is 0, this affects the formula

Proof Notice $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$

$$= a \cdot \lim_{n\to\infty} \frac{1-r^n}{1-r} = \frac{a}{1-r}$$ when $|r|<1$

Also $\lim_{n\to\infty} S_n$ does not exist when $|r| \geq 1$.

Example (Drug dosage) Find

(1) Q_{40}

(2) $\lim_{n\to\infty} Q_n$

We know $a + ar + \cdots + ar^{n-1} = \dfrac{a(1-r^n)}{1-r}$.

(1) We also know

$$Q_n = 250 + 250(0.04) + \cdots + 250(0.04)^{n-1}$$

so that $Q_{40} = \underbrace{250}_{=a} + 250(\underbrace{0.04}_{=r}) + \cdots + 250(0.04)^{\overbrace{39}^{=n-1}}$

$$= \frac{250(1-(0.04)^{40})}{1-0.04}$$

(2) $\lim_{n\to\infty} Q_n = \lim_{n\to\infty} \dfrac{250(1-(0.04)^n)}{1-0.04}$

$$= \frac{250}{1-0.04}$$