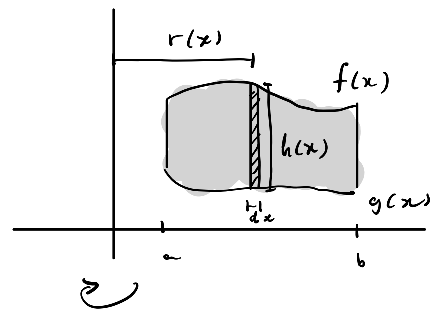
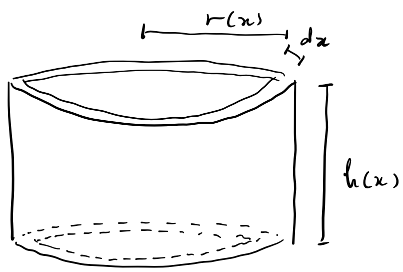
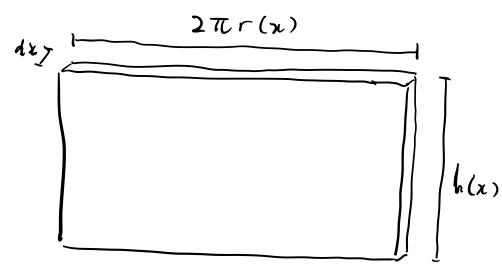
7.3 Shell Method

We introduce one more method for finding volumes of revolution called the shell method.





shell formed by revolving slice



shell, but rolled out after cutting it along a side

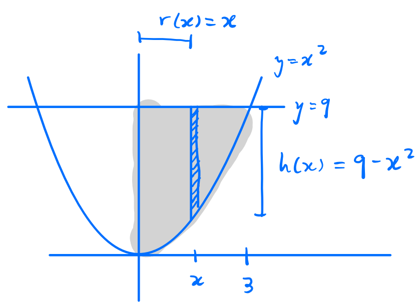
$\boxed{\text{Volume of shell is } 2\pi r(x)h(x)}$

Suppose we revolve the region between $y=f(x)$, $y=g(x)$, $x=a$, $x=b$ with vertical slices around the y-axis. We no longer get disks/washers. We get cylindrical shells.

Example Region bounded by $y=x^2$, $y=9$ in first quadrant is revolved around ① y-axis, ② $x=4$, ③ x-axis, ④ $y=-1$

Set up integrals for volume using shell method.

①



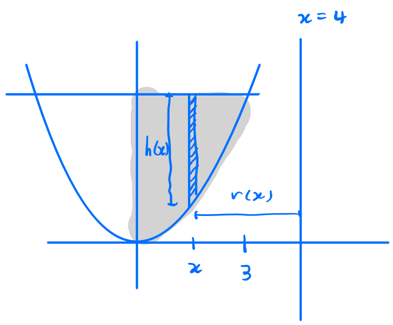
- use vertical slices to get shells
- volume of a shell is

$$2\pi r(x) h(x) = 2\pi x\,(9-x^2)\,dx$$

- total volume is

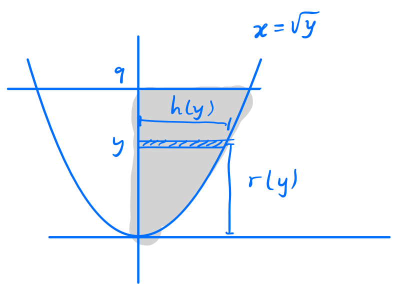
$$\int_0^3 2\pi x\,(9-x^2)\,dx$$

②



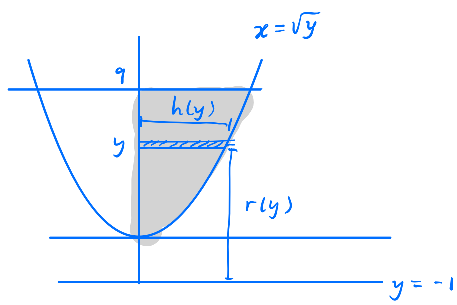
- use vertical slices again
- $h(x) = 9-x^2$, $r(x) = 4-x$
- volume $= \displaystyle\int_0^3 2\pi\,(4-x)\,(9-x^2)\,dx$

③



- use horizontal slices to get shells
- $h(y) = \sqrt{y}$, $r(y) = y$
- volume $= \displaystyle\int_0^9 2\pi\, y\sqrt{y}\, dy$

④



- use horizontal slices again
- $h(y) = \sqrt{y}$, $r(y) = y-(-1) = y+1$
- volume $= \displaystyle\int_0^9 2\pi\,(y+1)\sqrt{y}\, dy$

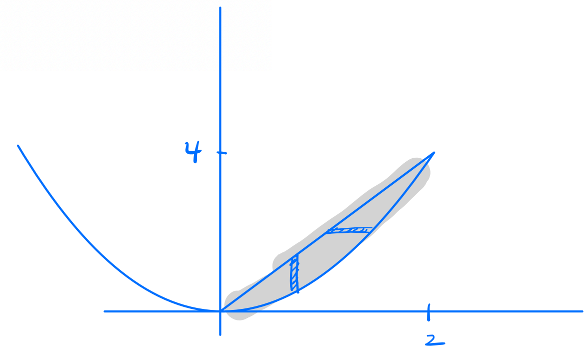
Problem 1. Use the shell method to set up an integral to find the volume of the solid of revolution formed by revolving the region bounded between $y = 2x$ and $y = x^2$ around each of the axes below.

a. $x = 0$.

b. $x = -1$.

c. $y = 0$.

d. $y = \cancel{3}$ 5



(a) $r(x) = x$, $h(x) = 2x - x^2$

$$\int_0^2 2\pi x (2x - x^2)\, dx$$

(b) $r(x) = x - (-1) = x + 1$, $h(x) = 2x - x^2$

$$\int_0^2 2\pi (x+1)(2x - x^2)\, dx$$

(c) $r(y) = y$, $h(y) = \sqrt{y} - y/2$

$$\int_0^4 2\pi y (\sqrt{y} - y/2)\, dy$$

(d) $r(y) = 5 - y$, $h(y) = \sqrt{y} - y/2$

$$\int_0^4 2\pi (5-y)(\sqrt{y} - y/2)\, dy$$

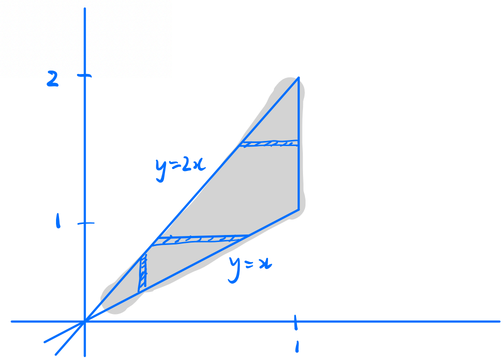
Problem 2. Use the shell method to set up an integral to find the volume of the solid of revolution formed by revolving the region bounded between $y = x, y = 2x$ and $x = 1$ around each of the axes below.

a. $x = 0$

b. $x = 1$

c. $y = 0$

d. $y = -2$



(a) $r(x) = x, \quad h(x) = 2x - x = x$

$$\int_0^1 2\pi x^2 \, dx$$

(b) $r(x) = 1 - x, \quad h(x) = 2x - x = x$

$$\int_0^1 2\pi (1-x) x \, dx$$

(c) $r(y) = y, \quad h(y) = y - y/2 = y/2 \quad$ when $0 \le y \le 1$

$r(y) = y, \quad h(y) = 1 - y/2 \quad$ when $1 \le y \le 2$

$$\int_0^1 2\pi \, y^2/2 \, dy + \int_1^2 2\pi y (1 - y/2) \, dy$$

(d) $r(y) = y + 2, \quad h(y) = y - y/2 = y/2 \quad$ when $0 \le y \le 1$

$r(y) = y + 2, \quad h(y) = 1 - y/2 \quad$ when $1 \le y \le 2$

$$\int_0^1 2\pi (y+2) \, y/2 \, dy + \int_1^2 2\pi (y+2)(1 - y/2) \, dy$$

Problem 3. Fill in each entry of the table below with two pieces of information: the orientation of slices (vertical or horizontal) and the variable of integration (x or y) to be used given the method and orientation of the axis of revolution.

	Disk/washer method	Shell method
Horizontal axis	vertical, x	horizontal, y
Vertical axis	horizontal, y	vertical, x