7.5 Work

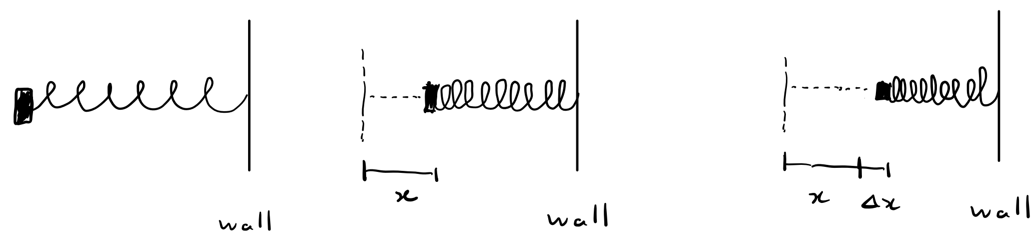
In physics, work is quantity W given by a force F applied to move an object a distance d. The force must be applied in the direction of motion and if it's a constant force $W = Fd$.

Example Find work done to lift a 3 kg stone 5 meters up off the ground.

$$W = (\underbrace{3 \cdot 9.8}_{\substack{F = mg \\ \text{(force due to gravity)}}})(\underbrace{5}_{\text{distance}})$$

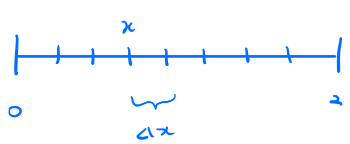
We are often interested in examples where the force is not constant.

Hooke's Law says the force F required to compress a spring x meters from its equilibrium position is given by $F(x) = kx$ where k is a given constant specific to the makeup of the spring.



equilibrium, $x=0$ compressed x meters   compressed $x+\Delta x$ meters

Example: Consider a spring with $k = 8$ N/m. Find work to compress the spring 2 meters.



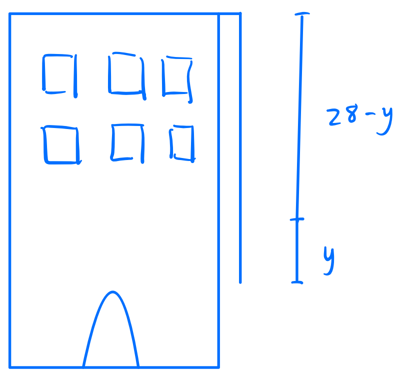
Divide distance into increments of Δx

Work to compress from x to $x+\Delta x$ meters from equilibrium $= \underbrace{kx}_{F}\underbrace{\Delta x}_{d}$

$$\text{Total work} = \int_0^2 \underbrace{kx}_{F}\,\underbrace{dx}_{d} = \int_0^2 8x\,dx = 4x^2\Big|_0^2 = 16 \text{ (Joules)}$$

Example A 28 m chain with mass density 2 kg/m is hanging off the roof of a building.

Find work done to pull the chain onto the top of the building.



mass of segment of chain of length Δy

$= 2\Delta y$

distance the segment y meters from dangling end must travel

$= 28 - y$

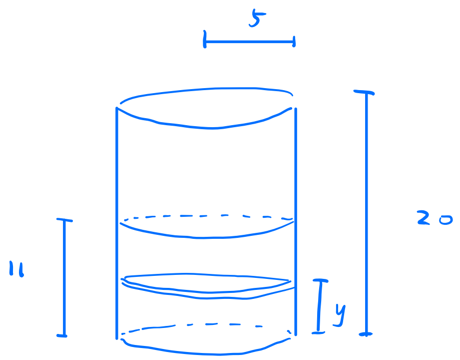
Work done in lifting this segment

$$= \underbrace{(2\Delta y)(9.8)}_{F=ma}\ \underbrace{(28-y)}_{d} = 19.6(28-y)\Delta y$$

Total work for whole chain

$$\int_0^{28} 19.6(28-y)\,dy = 7683.2 \text{ J.}$$

Example A cylindrical tank (height 20 meters, radius 5 meters) is filled to a height of 11 meters with a liquid with density 3 kg/m^3. Find work required to pump it to the top of tank.



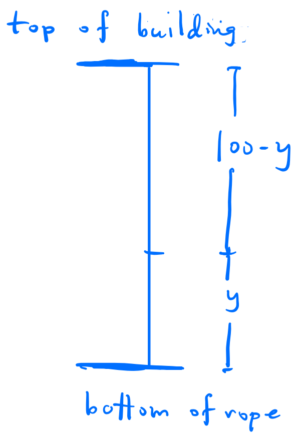
mass of slice of width Δy

$$= \underbrace{\left(\pi (5)^2 \Delta y\right)}_{\text{volume}} \underbrace{(3)}_{\text{density}} = 75\pi \Delta y$$

distance slice at height y must be pumped $= 20 - y$

$$\text{Total work} = \int_0^{11} \underbrace{(75\pi)(9.8)}_{F=ma} \underbrace{(20-y)}_{d} \, dy$$

Problem 1. A 100 meter rope of density 0.1 kilograms per meter is dangling off the side of a building. Find the work to lift the rope completely onto the top of the building.



work to lift segment y meters from bottom of rope

$$\underbrace{(0.1\Delta y \cdot 9.8)}_{F=ma}\ \underbrace{(100-y)}_{d}$$

$$\text{total work} = \int_0^{100} .98\,(100-y)\,dy$$

$$= \int_0^{100} (98 - 0.98y)\,dy$$

$$= 9800 - 0.49y^2 \Big|_0^{100}$$

$$= 9800 - 4900 = 4900$$

Problem 2. A force of 2 Newtons compresses a spring 5 meters.

a. Use this information to find the spring constant k.

b. Find the work done in compressing the spring.

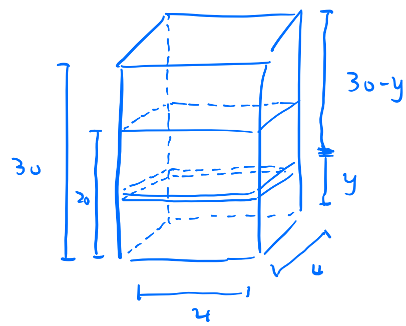
(a) $F(x) = kx$

$2 = k(5)$

$k = 0.4$

(b) $$\int_0^5 0.4x\,dx = 0.2x^2 \Big|_0^5 = 5$$

Problem 3. A tank with height 30 meters and square base with side length 4 meters is filled to a height of 20 meters with fluid. The fluid has density 2.2 kg per cubic meter. Set up an integral to find the work to pump the fluid to the top of the tank.



mass of slice

$$= \underbrace{2.2}_{\text{density}} \underbrace{(4)(4)(\Delta y)}_{\text{volume}} = 35.2\Delta y$$

work to lift slice at height y

$$\underbrace{(35.2\Delta y)(9.8)}_{F=ma} \underbrace{(30-y)}_{d}$$

$$\text{total work} = \int_0^{20} 9.8(35.2)(30-y)\,dy$$