# 8.3 Integral test

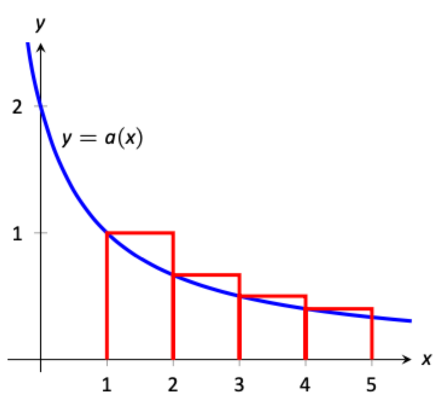
Consider a series $\sum_{n=1}^{\infty} a_n$ where

$a_n = a(n)$ and $a(x)$ is a continuous

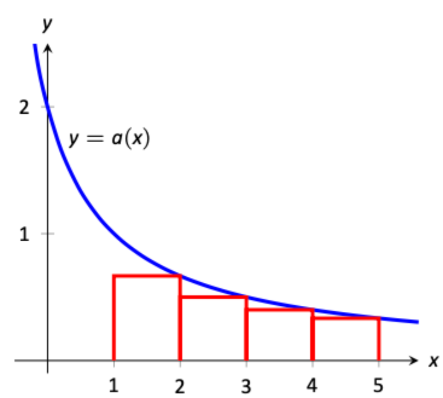
decreasing function with $a(x) \geq 0$ for all $x \geq 1$.

How can we relate the convergence/divergence

of $\sum_{n=1}^{\infty} a_n$ with that of $\int_1^{\infty} a(x)\,dx$ ?




$$\sum_{n=1}^{\infty} a_n > \int_1^{\infty} a(x)\,dx$$




$$\sum_{n=2}^{\infty} a_n < \int_1^{\infty} a(x)\,dx$$

$$\sum_{n=1}^{\infty} a_n < a_1 + \int_1^{\infty} a(x)\,dx$$

**Theorem** (Integral Test) Let $a(x)$ be a continuous, nonnegative, decreasing function on $[1,\infty)$, and let $a_n = a(n)$ for all $n \geq 1$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} a(x)\,dx$ converges.

**Examples** Use the integral test to determine convergence/divergence of the following series.

① $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p \neq 1$ ② $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ ③ $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^4}$

①
$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b x^{-p}\,dx$$
$$= \lim_{b\to\infty} \frac{1}{-p+1} x^{-p+1} \Big|_1^b$$
$$= \lim_{b\to\infty} \frac{1}{-p+1} \left(b^{-p+1} - 1\right)$$
$$= \begin{cases} \frac{-1}{-p+1} & \text{when } -p+1<0 \ (p>1) \\ \infty & \text{when } -p+1>0 \ (p<1) \end{cases}$$

converges when $p>1$, diverges when $p<1$

② $\int_1^\infty \frac{\ln x}{x^2}\,dx$

$u = \ln x \qquad dv = \frac{1}{x^2}\,dx$

$du = \frac{1}{x}\,dx \qquad v = \frac{-1}{x}$

$$= \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\,dx$$

$$= \lim_{b\to\infty} -\frac{1}{x}\ln x \Big|_1^b + \int_1^b \frac{1}{x^2}\,dx$$

$$= \lim_{b\to\infty} \frac{-\ln b}{b} + -x^{-1}\Big|_1^b$$

$$= \lim_{b\to\infty} \frac{-\ln b}{b} - \frac{1}{b} + 1 = 1$$ , converges

③ $\int_2^\infty \frac{1}{x(\ln x)^4}\,dx$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^4}\,dx$$

$$= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{u^4}\,du$$

$$= \lim_{b\to\infty} -\frac{3}{u^3}\Big|_{\ln 2}^{\ln b} = \lim_{b\to\infty} \frac{3}{(\ln 2)^3} - \frac{3}{(\ln b)^3} = \frac{3}{(\ln 2)^3}$$

Converges.

**Theorem** (L'Hôpital's Rule) If $f$ and $g$ are differentiable and $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \pm\infty$ (or both $0$), then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$

**Example** $\displaystyle\lim_{x \to \infty} \frac{\ln x}{x}$

$\displaystyle\lim_{x \to \infty} \ln x = \lim_{x \to \infty} x = \infty$, so

$$\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = \lim_{x \to \infty} \frac{1}{x} = 0.$$

**Problem 1.** Use the integral test to determine the convergence or divergence of the following series.

a. $\sum_{n=1}^{\infty} \frac{1}{n}$

b. $\sum_{n=1}^{\infty} e^{-n}$

c. $\sum_{n=1}^{\infty} \frac{2n}{(1+n^2)^3}$

d. $\sum_{n=1} \frac{\ln n}{n}$

(a)

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b \to \infty} \int_1^b \frac{1}{x}\,dx$$

$$= \lim_{b \to \infty} \ln|x| \Big|_1^b$$

$$= \lim_{b \to \infty} \ln b = \infty, \text{ diverges}$$

(b)

$$\int_1^{\infty} e^{-x}\,dx = \lim_{b \to \infty} \int_1^b e^{-x}\,dx$$

$$= \lim_{b \to \infty} -e^{-x} \Big|_1^b$$

$$= \lim_{b \to \infty} e^{-1} - e^{-b} = e^{-1}, \text{ converges}$$

(c)

$$\int_1^{\infty} \frac{2x}{(1+x^2)^3}\,dx$$

$$= \lim_{b \to \infty} \int_1^b \frac{2x}{(1+x^2)^3}\,dx \qquad u = 1+x^2$$

$$du = 2x\,dx$$

$$= \lim_{b \to \infty} \int_2^{1+b^2} \frac{1}{u^3}\,du$$

$$= \lim_{b \to \infty} -\frac{1}{2}u^{-2} \Big|_2^{1+b^2} = \lim_{b \to \infty} \frac{1}{8} - \frac{1}{2(1+b^2)^2} = \frac{1}{8}, \text{ converges}$$

(d) $$\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x}\,dx \qquad u = \ln x \quad dv = \frac{1}{x}\,dx$$

$$du = \frac{1}{x} \qquad v = \ln x$$

$$= \lim_{b\to\infty} (\ln x)^2\Big|_1^b - \int_1^b \frac{\ln x}{x}\,dx$$

$$= \frac{1}{2}\lim_{b\to\infty} (\ln x)^2\Big|_1^b = \frac{1}{2}\lim_{b\to\infty} (\ln b)^2 = \infty, \text{ diverges}$$

**Problem 2.** True or false: $\sum_{n=1}^{\infty} \frac{1}{n^3}$ and $\int_1^\infty \frac{1}{x^3}\,dx$ both converge to 1/2.

False, $\int_1^\infty \frac{1}{x^3}dx$ converges to $\frac{1}{2}$, but $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges to a value smaller than $1 + \int_1^\infty \frac{1}{x^3}dx = \frac{3}{2}$

**Problem 3.** Explain why the integral test cannot be applied to the following series.

a. $\sum_{n=1}^{\infty} n^2$

b. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

c. $\sum_{n=1}^{\infty} e^{-n}\sin n$

(a) $x^2$ not decreasing for $x \geq 1$

(b) $\frac{(-1)^x}{x}$ not non-negative for all $x \geq 1$

(c) $e^{-x}\sin x$ not non-negative, not decreasing for all $x \geq 1$