Problem 1. Evaluate the following integrals.

a. $\int \frac{dx}{5-3x}$

b. $\int \frac{(\ln x)^2}{x}\, dx$

c. $\int x(2x^2+5)^8\, dx$

d. $\int \sqrt{x}\sin(1+x^{3/2})\, dx$

e. $\int_1^2 \frac{e^{1/x}}{x^2}\, dx$

a) $\int \frac{dx}{5-3x}$ $u = 5-3x$

$du = -3dx$

$= -\frac{1}{3}\int \frac{1}{u}\, du$ $-\frac{1}{3}du = dx$

$= -\frac{1}{3}\ln|u| + C = -\frac{1}{3}\ln|5-3x| + C$

b) $\int \frac{(\ln x)^2}{x}\, dx$ $u = \ln x$

$du = \frac{1}{x}\, dx$

$= \int u^2\, du$

$= \frac{1}{3}u^3 + C = \frac{1}{3}\ln^3 x + C$

c) $\int x(2x^2+5)^8\, dx$ $u = 2x^2+5$

$du = 4x\,dx$

$= \frac{1}{4}\int u^8\, du$ $\frac{1}{4}du = x\,dx$

$= \frac{1}{36}u^9 + C = \frac{1}{36}(2x^2+5)^9 + C$

d) $\int \sqrt{x}\sin(1+x^{3/2})\, dx$ $u = 1+x^{3/2}$

$du = \frac{3}{2}x^{1/2}\, dx$

$= \frac{2}{3}\int \sin u\, du$ $\frac{2}{3}du = x^{1/2}dx$

$= -\frac{2}{3}\cos u + C = -\frac{2}{3}\cos(1+x^{3/2}) + C.$

e) $\int_1^2 \frac{e^{1/x}}{x^2}\, dx$ $u = \frac{1}{x}$

$du = -\frac{1}{x^2}dx$

$= -\int_1^{1/2} e^u\, du$ $-du = \frac{1}{x^2}dx$

$= \int_{1/2}^1 e^u\, du = e^u\Big|_{1/2}^1 = e - e^{1/2}$

Problem 2. Evaluate the following integrals

a. $\int x \cos 5x \, dx$

b. $\int_1^e x^3 \ln x \, dx$

c. $\int (x^2 + 2x) \cos x \, dx$

a) $\int x \cos(5x) dx$ $\quad u = x \quad dv = \cos(5x)dx$

$du = dx \quad v = \frac{1}{5}\sin(5x)$

$= \frac{1}{5} x \sin(5x) - \frac{1}{5}\int \sin(5x) dx$

$= \frac{1}{5} x \sin(5x) + \frac{1}{25}\cos(5x) + C$

b) $\int_1^e x^3 \ln x \, dx$ $\quad u = \ln x \quad dv = x^3 dx$

$du = \frac{1}{x} dx \quad V = \frac{1}{4}x^4$

$= \frac{1}{4} x^4 \ln x \Big|_1^e - \int_1^e \frac{1}{4} x^3 dx$

$= \frac{1}{4} e^4 - \frac{1}{16} x^4 \Big|_1^e$

$= \frac{1}{4} e^4 + \frac{1}{16} - \frac{1}{16} e^4$

$= \frac{3}{16} e^4 + \frac{1}{16}$

c) $\int (x^2 + 2x) \cos x \, dx$ $\quad u = x^2 + 2x \quad dv = \cos x dx$

$du = (2x + 2)dx \quad v = \sin x$

$= (x^2 + 2x)\sin x - \int (2x+2)\sin x dx$

$u = 2x + 2 \quad dv = \sin x dx$

$du = 2dx \quad V = -\cos x$

$= (x^2+2x)\sin x - \left(-(2x+2)\cos x + \int 2\cos x \, dx\right)$

$= (x^2+2x)\sin x + (2x+2)\cos x - 2\sin x + C$

Problem 3. Evaluate the following integrals.

a. $\displaystyle\int \frac{x^2+x+1}{x^3+x^2-2x}\,dx$

b. $\displaystyle\int \frac{10}{(x-1)(x^2+9)}\,dx$

c. $\displaystyle\int \frac{x^2-5x+16}{(2x+1)(x-2)^2}\,dx$

a) $\displaystyle\int \frac{x^2+x+1}{x^3+x^2-2x}\,dx$

$$\frac{x^2+x+1}{x(x^2+x-2)} = \frac{x^2+x+1}{x(x+2)(x-1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-1}$$

$$x^2+x+1 = A(x+2)(x-1) + Bx(x-1) + Cx(x+2)$$

$\underline{x=-2}$ $\quad 3 = 6B \Rightarrow B = \frac{1}{2}$

$\underline{x=0}$ $\quad 1 = -2A \Rightarrow A = -\frac{1}{2}$

$\underline{x=1}$ $\quad 3 = 3C \Rightarrow C = 1$

$$\int \frac{-\frac{1}{2}}{x}\,dx + \int \frac{\frac{1}{2}}{x+2}\,dx + \int \frac{1}{x-1}\,dx$$

$$= -\frac{1}{2}\ln|x| + \frac{1}{2}\ln|x+2| + \ln|x-1| + c$$

b) $$\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9}$$

$$10 = A(x^2+9) + (Bx+C)(x-1)$$

$\underline{x=1}$ $\quad 10 = 10A, \quad A=1$

$\underline{x=0}$ $\quad 10 = 9 - C, \quad C=-1$

$\underline{x=2}$ $\quad 10 = 13 + 2B - 1, \quad B=-1$

$$\int \frac{1}{x-1}\,dx + \int \frac{-x-1}{x^2+9}\,dx$$

$$= \ln|x-1| + \int \frac{-x}{x^2+9}\,dx + \int \frac{-1}{x^2+9}\,dx$$

$u = x^2+9$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= \ln|x-1| - \frac{1}{2}\int \frac{1}{u}\,du - \frac{1}{9}\int \frac{1}{\frac{x^2}{9}+1}\,dx$$

$$= \ln|x-1| - \frac{1}{2}\ln|x^2+9| - \frac{1}{9}\int \frac{1}{(\frac{x}{3})^2+1}\,dx$$

$u = \frac{x}{3}$

$du = \frac{1}{3}dx$

$3du = dx$

$$= \ln|x-1| - \frac{1}{2}\ln|x^2+9| - \frac{1}{3}\int \frac{1}{u^2+1}\,du$$

$$= \ln|x-1| - \frac{1}{2}\ln|x^2+9| - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

c) $$\int \frac{x^2-5x+16}{(2x+1)(x-2)^2}\,dx$$

$$\frac{x^2-5x+16}{(2x+1)(x-2)^2} = \frac{A}{2x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$x^2-5x+16 = A(x-2)^2 + B(2x+1)(x-2) + C(2x+1)$$

$\underline{x=-\frac{1}{2}}$ $$\frac{1}{4} + \frac{5}{2} + 16 = \frac{25A}{4} \Rightarrow A = 3$$

$\underline{x=2}$ $$10 = 5C \Rightarrow C = 2$$

$\underline{x=0}$ $$16 = 4A - 2B + C$$

$$= 12 - 2B + 2$$

$$B = -1$$

$$\int \frac{3}{2x+1}\,dx + \int \frac{-1}{x-2}\,dx + \int \frac{2}{(x-2)^2}\,dx$$

$u = 2x+1$

$du = 2dx$

$$= \frac{3}{2}\int \frac{1}{u}\,du - \ln|x-2| - 2(x-2)^{-1} + C$$

$$= \frac{3}{2}\ln|2x+1| - \ln|x-2| - \frac{2}{x-2} + C$$

Problem 4. Evaluate the following improper integrals or show that they diverge.

a. $\int_3^\infty \frac{1}{2(x+5)^2}\,dx$

b. $\int_{-1}^4 \frac{1}{(x+1)^{1/3}}\,dx$

c. $\int_{-3}^3 \frac{1}{x^{2/3}}\,dx$

d. $\int_0^\infty \frac{1}{(x-1)^4}\,dx$

a)
$$\int_3^\infty \frac{1}{2(x+5)^2}\,dx$$
$$= \lim_{b\to\infty} \int_3^b \frac{1}{2(x+5)^2}\,dx$$
$$= \lim_{b\to\infty} -\frac{1}{2}(x+5)^{-1}\Big|_3^b$$
$$= \lim_{b\to\infty} \left(\frac{1}{2}(8)^{-1} - \frac{1}{2}\frac{1}{b+5}\right)$$
$$= \frac{1}{16}$$

b)
$$\int_{-1}^4 \frac{1}{(x+1)^{1/3}}\,dx$$
$$= \lim_{a\to -1^+} \int_a^4 \frac{1}{(x+1)^{1/3}}\,dx$$
$$= \lim_{a\to -1^+} \frac{3}{2}(x+1)^{2/3}\Big|_a^4$$
$$= \lim_{a\to -1^+} \frac{3}{2}(5)^{2/3} - \frac{3}{2}(a+1)^{2/3}$$
$$= \frac{3}{2}(5)^{2/3}$$

c)
$$\int_{-3}^3 \frac{1}{x^{2/3}}\,dx = \int_{-3}^0 \frac{1}{x^{2/3}}\,dx + \int_0^3 \frac{1}{x^{2/3}}\,dx$$
$$= \lim_{b\to 0^-} \int_{-3}^b \frac{1}{x^{2/3}}\,dx + \lim_{a\to 0^+} \int_a^3 \frac{1}{x^{2/3}}\,dx$$
$$= \lim_{b\to 0^-} 3x^{1/3}\Big|_{-3}^b + \lim_{a\to 0^+} 3x^{1/3}\Big|_a^3$$
$$= \lim_{b\to 0^-} 3b^{1/3} - 3(-3)^{1/3} + \lim_{a\to 0^+} 3(3)^{1/3} - 3a^{1/3} = 6(3)^{1/3}$$

d) $\displaystyle\int_0^\infty \frac{1}{(x-1)^3}\,dx$

$$= \int_0^1 \frac{1}{(x-1)^3}\,dx + \int_1^2 \frac{1}{(x-1)^3}\,dx + \int_2^\infty \frac{1}{(x-1)^3}\,dx$$

$$\int_0^1 \frac{1}{(x-1)^3}\,dx = \lim_{b\to 1^-} \int_0^b \frac{1}{(x-1)^3}\,dx$$

$$= \lim_{b\to 1^-} -\frac{1}{2}(x-1)^{-2}\Big|_0^b$$

$$= \lim_{b\to 1^-} \frac{1}{2}(-1)^{-2} - \frac{1}{2}(b-1)^{-2}$$

$$= \frac{1}{2} - \frac{1}{2}\lim_{b\to 1^-} \frac{1}{(b-1)^2}$$

$= -\infty$, so the whole integral

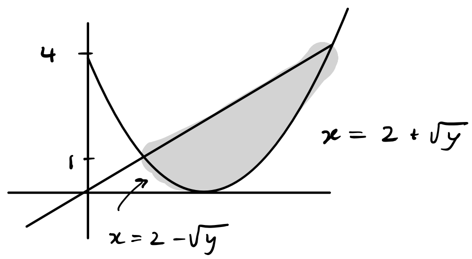
diverges since one term diverges.

Problem 5. Sketch the region enclosed by the given curves and then set up integrals for the unsigned area of the region in two ways: with respect to x and with respect to y.

a. $y = (x-2)^2, \quad y = x$

b. $y = x^2 - 2x, \quad y = x + 4$

a)



Intersection points:

$$x = (x-2)^2$$

$$x = x^2 - 4x + 4$$

$$0 = x^2 - 5x + 4$$

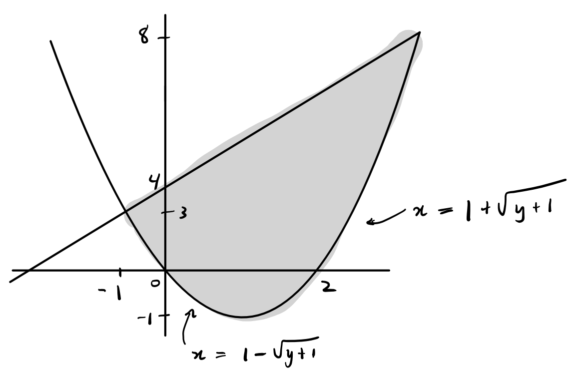
$$= (x-4)(x-1)$$

$$x = 1, 4$$

$$\int_1^4 \left(x - (x-2)^2\right) dx$$

$$= \int_0^1 \left(2+\sqrt{y} - (2-\sqrt{y})\right) dy + \int_1^4 \left(2+\sqrt{y} - y\right) dy$$

b)



Intersection points

$$x^2 - 2x = x + 4$$

$$\Leftrightarrow \quad x^2 - 3x - 4 = 0$$

$$\Leftrightarrow \quad (x-4)(x+1) = 0$$

$$x = -1, 4$$

$$\text{Area} = \int_{-1}^{4} \left[(x+4) - (x^2 - 2x)\right] dx$$

$$= \int_{-1}^{3} \left(\left(1+\sqrt{y+1}\right) - \left(1 - \sqrt{y+1}\right)\right) dy + \int_3^8 \left(1 + \sqrt{y+1} - (y-4)\right) dy$$

Problem 6. Sketch the region bounded by the given curves and then set up integrals to find the volume obtained by rotating the region about the given axis. Do this in two ways: using the disk/washer method and using the shell method.

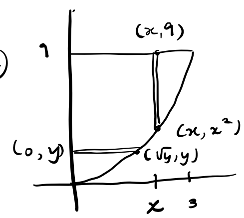
a. $y = x^2, \quad y = 9, \quad x = 0;$ about the x-axis

b. $y = \sqrt{x}, \quad y = 0, \quad x = 1;$ about $x = -1$

c. $y = x^3, \quad y = 0, \quad x = 2;$ about $y = -2$

d. $y = \ln x, \quad y = 1, \quad y = 2, \quad x = 0;$ about the y-axis

(a)



Washer method

$$V = \int_0^3 (\pi R_1^2 - \pi R_2^2)\,dx$$

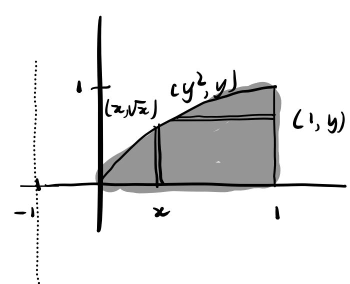
$$= \int_0^3 \left(\pi(9)^2 - \pi(x^2)^2\right)dx$$

shell method

$$V = \int_0^9 2\pi r h\,dy$$

$$= \int_0^9 2\pi(y)(\sqrt{y})\,dy$$

(b)



washer method

$$\int_0^1 (\pi R_1^2 - \pi R_2^2)\,dy$$

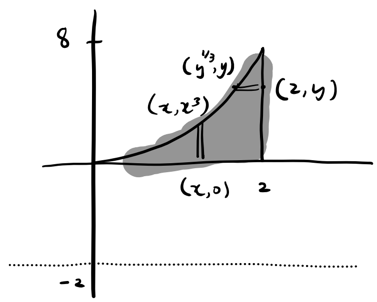
$$= \int_0^1 \left(\pi(2)^2 - \pi\left(y^2+1\right)^2\right)dy$$

shell method

$$\int_0^1 2\pi r h\,dx$$

$$= \int_0^1 2\pi\left(x+1\right)(\sqrt{x})\,dx$$

(c)



shell method

$$V = \int_0^8 2\pi r h \, dy$$

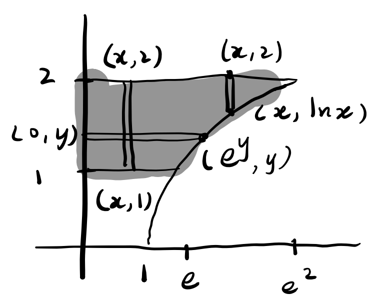
$$= \int_0^8 2\pi (y+2)(2 - y^{1/3}) \, dy$$

washer method

$$V = \int_0^2 (\pi R_1^2 - \pi R_2^2) \, dx$$

$$= \int_0^2 \left(\pi (x^3+2)^2 - \pi (2)^2\right) dx$$

(d)



disc method

$$\int_1^2 \pi R^2 \, dy = \int_1^2 \pi (e^y)^2 \, dy$$

shell method

$$\int_0^e 2\pi r h \, dx + \int_e^{e^2} 2\pi r h \, dx$$

$$= \int_0^e 2\pi (x)(1) \, dx + \int_e^{e^2} 2\pi (x)(2 - \ln x) \, dx$$