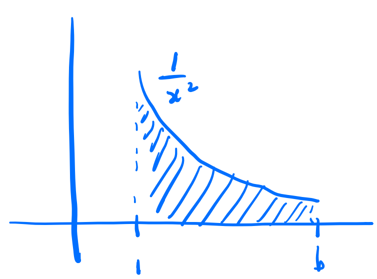
§7.6 Improper integrals

An improper integral is a definite integral where a limit of integration is either

① infinite

or ② a point where the integrand is unbounded.

Example $\int_1^\infty \frac{1}{x^2}\,dx$ We've seen integrals like this earlier but let's talk about them in more detail:



We know how to work with definite integrals over finite intervals

$$\int_1^b \frac{1}{x^2}\,dx$$

$$= \int_1^b x^{-2}\,dx = -x^{-1}\Big|_1^b$$

$$= 1 - \frac{1}{b}$$

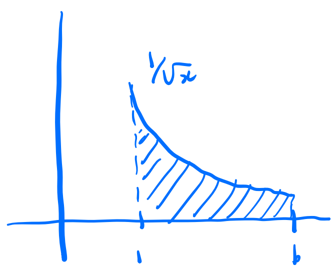
So we'll take $b \to \infty$ to get our improper integral

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left(1 - \frac{1}{b}\right)$$

$$= 1$$

<u>Definition</u> $\int_a^\infty f(x)\,dx$ <u>converges</u> if $\lim_{b\to\infty} \int_a^b f(x)\,dx$ is finite. If not, we say the integral <u>diverges</u>.

<u>Example</u> Does $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ converge?



$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty} \int_1^b x^{-1/2}\,dx$$

$$= \lim_{b\to\infty} 2x^{1/2}\Big|_1^b$$

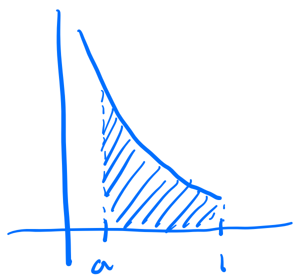
$$= 2\lim_{b\to\infty} \left(b^{1/2} - 1\right)$$

$$= \infty$$

No, it does not converge.

<u>Example</u> Does $\int_0^1 \frac{1}{\sqrt{x}}$ converge?

How about $\int_0^1 \frac{1}{x^2}$? What's different about these? How do we handle them?



Here, the function grows unboundedly near 0. So we'll cut off the integral at a and let $a \to 0^+$

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{a \to 0^+} \int_a^1 x^{-1/2}\,dx$$

$$= \lim_{a \to 0^+} 2x^{1/2} \Big|_a^1$$

$$= 2 \lim_{a \to 0^+} (1 - a^{1/2})$$

$$= 2 \quad \text{converges}$$

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{a \to 0^+} \int_a^1 x^{-2}\,dx$$

$$= \lim_{a \to 0^+} -x^{-1} \Big|_a^1$$

$$= \lim_{a \to 0^+} (a^{-1} - 1) = \infty \quad \text{diverges.}$$