8.6 Power series

A <u>power series</u> is an infinite series that involves a variable x and has the following form:

$$\sum_{n=0}^{\infty} a_n x^n \quad \text{or} \quad \sum_{n=0}^{\infty} a_n (x-c)^n$$

where $c, a_0, a_1, a_2, \ldots$ are given constants.

<u>Intuition</u>: it's like a polynomial with infinitely many terms.

<u>Goal</u> determine which values of x can be plugged in so that we get a convergent series (ie. determine the domain of this infinite polynomial) for examples.

<u>Example</u> For which values of x does the following power series converge:

$$\sum_{n=0}^{\infty} \frac{x^n}{2^n} \quad (\text{does } x=1 \text{ work? } x=3\text{?})$$

<u>$x=1$</u> $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ converges

since it's a geometric series with $r=\frac{1}{2}$

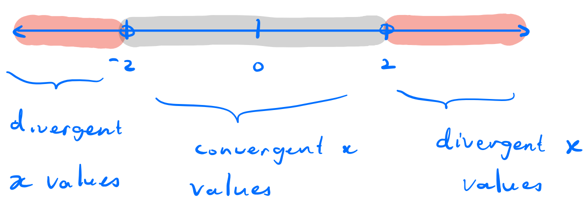
<u>$x=3$</u> $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$, diverges: $r=\frac{3}{2}>1$

General x $\quad \sum_{n=0}^{\infty} \frac{x^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$ converges

when $\left|\frac{x}{2}\right| < 1$, diverges when $\left|\frac{x}{2}\right| \geq 1$

$$\left|\frac{x}{2}\right| < 1 \iff -1 < \frac{x}{2} < 1 \iff -2 < x < 2$$

$$\left|\frac{x}{2}\right| \geq 1 \iff \frac{x}{2} \geq 1 \text{ or } \frac{x}{2} \leq -1 \iff x \geq 2 \text{ or } x \leq -2$$



Terminology $(-2, 2)$ is called the interval of convergence of this power series. And $R = 2$ is called the radius of convergence

Remark When this power series converges, it represents the function $f(x) = \frac{1}{1 - (\frac{x}{2})}$ (from geometrics series formula).

Example Find the interval of convergence and radius of convergence for the following power series.

① $\sum_{n=1}^{\infty} (-1)^n \frac{(x-1)^n}{n \cdot 3^n}$ ② $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

① We use the ratio test, treating x like a constant:

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-1)^{n+1}}{(n+1)\,3^{n+1}} \cdot \frac{n \cdot 3^n}{(-1)^n (x-1)^n} \right|$$

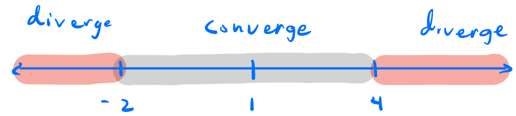
$$= \lim_{n\to\infty} \left|\frac{x-1}{3}\right| \cdot \frac{n}{n+1} = \left|\frac{x-1}{3}\right|$$

Series converges when $L<1$, diverges when $L>1$ and test is inconclusive when $L=1$.

$$L<1 \iff \left|\frac{x-1}{3}\right| < 1 \iff -1 < \frac{x-1}{3} < 1$$
$$\iff -3 < x-1 < 3$$
$$\iff -2 < x < 4$$

$$L>1 \iff \left|\frac{x-1}{3}\right| > 1 \iff \frac{x-1}{3} > 1 \text{ or } \frac{x-1}{3} < -1$$
$$\iff x-1 > 3 \text{ or } x-1 < -3$$
$$\iff x > 4 \text{ or } x < -2$$

$$L=1 \iff \left|\frac{x-1}{3}\right| = 1 \iff \frac{x-1}{3} = \pm 1$$
$$\iff x-1 = \pm 3 \iff x = -2, 4$$



What happens at endpoints? Use another test:

$\underline{x=-2}$

$$\sum_{n=1}^{\infty} (-1)^n \frac{(x-1)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} (-1)^n \frac{(-3)^n}{n \cdot 3^n}$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{(-1)^n \cdot 3^n}{n \cdot 3^n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges, } p=1$$

$\underline{x=4}$

$$\sum_{n=1}^{\infty} (-1)^n \frac{(x-1)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n \cdot 3^n}$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

converges by alternating series test.

Interval convergence: $(-2, 4]$ Radius of convergence: $R = 3$

don't include endpoint include endpoint

(2) Use ratio test:

$$L = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} |x| \cdot \frac{1}{n+1} = 0 \quad \text{for any } x$$

So $L < 1$ for all real numbers x.

Interval of convergence: $(-\infty, \infty)$

Radius of convergence: $R = \infty$.

Problem. For each of the following power series, find the interval and radius of convergence.

a. $\sum_{n=1}^{\infty} \frac{(5x)^n}{n}$

b. $\sum_{n=1}^{\infty} (-1)^n \frac{(x-2)^n}{\sqrt{n}5^n}$

c. $\sum_{n=1}^{\infty} (-1)^n \frac{(x+3)^n}{n^2 2^n}$

d. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

e. $\sum_{n=0}^{\infty} n! x^n$

(a)

$$\lim_{n\to\infty} \left| \frac{\frac{(5x)^{n+1}}{n+1}}{\frac{(5x)^n}{n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(5x)^{n+1}}{(5x)^n} \cdot \frac{n}{n+1} \right|$$

$$= |5x|$$

By the ratio test, the series converges when $|5x| < 1$ or $|x| < \frac{1}{5}$ or $-\frac{1}{5} < x < \frac{1}{5}$.

We must test $x = \pm\frac{1}{5}$ separately.

When $x = -\frac{1}{5}$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges

and $x = \frac{1}{5}$, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

So the interval of convergence is

$\left[-\frac{1}{5}, \frac{1}{5}\right)$, $R = \frac{1}{5}$

(b)

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{\sqrt{n+1}\cdot 5^{n+1}} \frac{\sqrt{n}\cdot 5^n}{(x-2)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x-2}{5} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} \right|$$

$$= \left| \frac{x-2}{5} \right|$$

By ratio test, the series converges when

$$\left| \frac{x-2}{5} \right| < 1$$

$$-1 < \frac{x-2}{5} < 1$$

$$-5 < x-2 < 5$$

$$-3 < x < 7$$

We'll test the endpoints separately:

$\underline{x = -3}$ $\sum_{n=1}^{\infty} (-1)^n \frac{(-5)^n}{\sqrt{n}\cdot 5^n} = \sum_{n=1}^{\infty} (-1)^n \frac{(-1)^n \cdot 5^n}{\sqrt{n}\cdot 5^n}$

$= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges $(p = \frac{1}{2})$.

$\underline{x = 7}$ $\sum_{n=1}^{\infty} (-1)^n \frac{5^n}{\sqrt{n}\cdot 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, converges by alt. series test.

Interval: $(-3, 7]$, $R = 5$

ⓒ $\lim_{n\to\infty} \left| \frac{(x+3)^{n+1}}{(n+1)^2 2^{n+1}} \cdot \frac{n^2 \cdot 2^n}{(x+3)^n} \right|$

$= \lim_{n\to\infty} \left| \frac{x+3}{2} \cdot \frac{n^2}{(n+1)^2} \right|$

$= \left| \frac{x+3}{2} \right|$

By ratio test,

the series will converge when

$\left| \frac{x+3}{2} \right| < 1$

$-1 < \frac{x+3}{2} < 1$

$-2 < x+3 < 2$

$-5 < x < -1$

$\underline{x = -5}$

$\sum_{n=1}^{\infty} (-1)^n \frac{(-5+3)^n}{n^2 \cdot 2^n}$

$= \sum_{n=1}^{\infty} (-1)^n \frac{(-1)^n \cdot 2^n}{n^2 \cdot 2^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

converges $p=2>1$.

$\underline{x = -1}$

$\sum_{n=1}^{\infty} (-1)^n \frac{(-1+3)^n}{n^2 \cdot 2^n} = \sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n^2 \cdot 2^n}$

$= \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n^2}$ converges by absolute convergence test.

interval: $[-5, -1]$ $R = 2$.

ⓓ $\lim_{n\to\infty} \left| \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right|$

$= \lim_{n\to\infty} \left| \frac{x^2}{(2n+3)(2n+2)} \right|$

$= 0 < 1$

So the series converges

for all x

interval: $(-\infty, \infty)$, $R = \infty$.

ⓔ $\lim_{n\to\infty} \left| \frac{(n+1)!\, x^{n+1}}{n!\, x^n} \right| = \lim_{n\to\infty} |x|\, n$

$= \begin{cases} 0 & \text{when } x = 0 \\ \infty & \text{when } x \neq 0 \end{cases}$

Series converges when $x = 0$ only.

Interval: $\{0\}$,

radius: $R = 0$