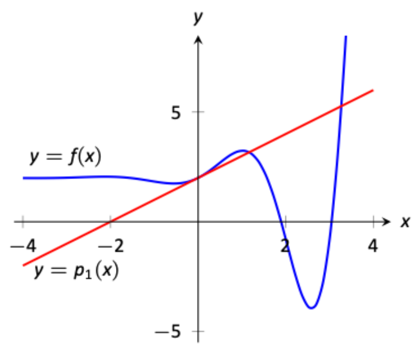
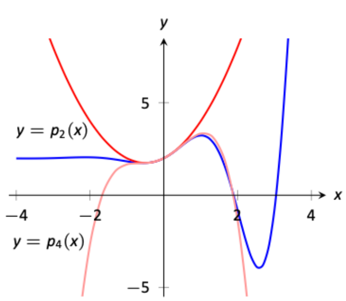
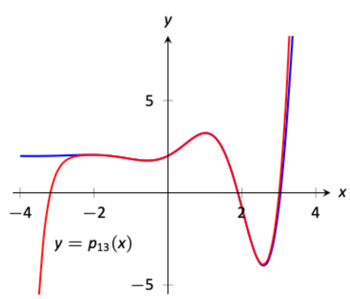
8.7 Taylor Polynomials

<u>Goal</u>: generalize the idea of tangent line approximation of a function f to quadratic, cubic, or nth degree polynomial approximations



<u>Question</u> what do $f(x)$ and its tangent line at 0, $p_1(x)$, have in common?





Quadratic, quartic, 13th degree polynomial approximations

Questions ① How should we define a quadratic polynomial so that $p_2(0) = f(0)$, $p_2'(0) = f'(0)$, $p_2''(0) = f''(0)$ (same value at 0, same slope at 0, same concavity at 0)?

Let $p_2(x) = c_0 + c_1 x + c_2 x^2$. Then

$$p_2'(x) = c_1 + 2c_2 x, \quad p_2''(x) = 2c_2.$$

We need to find c_0, c_1, c_2 so that these conditions are satisfied:

$$f(0) = p_2(0) = c_0$$

$$f'(0) = p_2'(0) = c_1$$

$$f''(0) = p_2''(0) = 2c_2 \Rightarrow c_2 = \frac{f''(0)}{2}$$

Therefore $p_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$

is our desired quadratic approximation.

② How do we find a quartic polynomial $p_4(x)$ so that $p_4(0) = f(0)$, $p_4'(0) = f'(0)$, $p_4''(0) = f''(0)$, $p_4'''(0) = f'''(0)$, $p_4^{(4)}(0) = f^{(4)}(0)$?

Let $p_4(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4$

$$p_4'(x) = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3$$

$$p_4''(x) = 2c_2 + 6c_3 x + 12 c_4 x^2$$

$$p_4'''(x) = 6c_3 + 24 c_4 x$$

$$p_4^{(4)}(x) = 24 c_4$$

We need to find c_0, c_1, c_2, c_3, c_4 so that:

$$f(0) = p(0) = c_0$$

$$f'(0) = p'(0) = c_1$$

$$f''(0) = p''(0) = 2c_2 \Rightarrow c_2 = \frac{f''(0)}{2}$$

$$f'''(0) = p'''(0) = 6c_3 \Rightarrow c_3 = \frac{f'''(0)}{6}$$

$$f^{(4)}(0) = p_4^{(4)}(0) = 24c_4 \Rightarrow c_4 = \frac{f^{(4)}(0)}{24}$$

So $p_4(x) = c_0 + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4$

Def Let f be a function whose first n derivatives exist at $x = c$.

① The Taylor polynomial of degree n at $x = c$ is

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

② The Maclaurin polynomial of degree n of f is

$$p_n(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

Example Find the Maclaurin polynomial of degree n of $f(x) = e^x$ and use $p_5(x)$ to approximate the value of e.

$$f^{(n)}(x) = e^x, \quad f^{(n)}(0) = e^0 = 1,$$

$$p_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n$$

$$e = f(1) \approx p_5(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} \approx 2.71667$$

Example Find the Maclaurin polynomial of degree 10 of $f(x) = \cos x$.

$f(x) = \cos x \qquad f(0) = 1 = f^{(4)}(0) = f^{(8)}(0)$

$f'(x) = -\sin x \qquad f'(0) = 0 = f^{(5)}(0) = f^{(9)}(0)$

$f''(x) = -\cos x \qquad f''(0) = -1 = f^{(6)}(0) = f^{(10)}(0)$

$f'''(x) = \sin x \qquad f'''(0) = 0 = f^{(7)}(0)$

$$P_{10}(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^6 - \frac{1}{10!}x^8$$

Example Find the Taylor polynomial at $x = 1$ of degree 6 of $f(x) = \ln x$.

$f(x) = \ln x \qquad f^{(4)}(x) = \frac{-6}{x^4}$

$f'(x) = \frac{1}{x} \qquad f^{(5)}(x) = \frac{24}{x^5}$

$f''(x) = \frac{-1}{x^2} \qquad f^{(6)}(x) = \frac{-120}{x^6}$

$f'''(x) = \frac{2}{x^3}$

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n}$$

$$f^{(n)}(1) = (-1)^{n+1}(n+1)!$$

$$P_n(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \cdots + \frac{f^{(n)}(1)}{n!}(x-1)^n$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \cdots + \frac{(-1)^{n-1}}{n}(x-1)^n$$

$$P_6(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6$$

Problem. Find the degree n Maclaurin polynomial for $f(x)$ in each of the following examples.

a. $f(x) = \sin x$, $n = 9$

b. $f(x) = 1/(1-x)$, $n = 5$

c. $f(x) = 1/(1+x)$, $n = 5$

d. $f(x) = e^{-x}$, $n = 5$

(a) $$p_9(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9$$

(b) $$p_5(x) = 1 + x + x^2 + x^3 + x^4 + x^5$$

(c) $$p_5(x) = 1 - x + x^2 - x^3 + x^4 - x^5$$

(d) $$p_5(x) = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5$$