§ 9.5 Power Series

<u>Example</u> Consider the power series

$$1 + 4x^2 + 16x^4 + 64x^6 + \cdots$$

Find its interval and radius of convergence.

Notice this series is $\sum_{n=0}^{\infty} 2^{2n} x^{2n}$. Using the Ratio Test, we have

$$L = \lim_{n\to\infty} \left| \frac{2^{2(n+1)} x^{2(n+1)}}{2^{2n} x^{2n}} \right|$$

$$= \lim_{n\to\infty} \frac{2^{2n+2}}{2^{2n}} \left| \frac{x^{2n+2}}{x^{2n}} \right|$$

$$= 4|x|^2$$

Then $L<1$ when $4|x|^2 < 1$

$$\iff |x|^2 < \frac{1}{4}$$

$$\iff |x| < \frac{1}{2}$$

$$\iff -\frac{1}{2} < x < \frac{1}{2}$$

When $x = \pm\frac{1}{2}$, the series is $\sum_{n=0}^{\infty} 1$ which diverges.

So the interval of convergence is $(-\frac{1}{2}, \frac{1}{2})$ and $R = \frac{1}{2}$.

Example Find the interval of convergence and radius of convergence for the following power series.

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Use ratio test:

$$L = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} |x| \cdot \frac{1}{n+1} = 0 \quad \text{for any } x$$

So $L < 1$ for all real numbers x.

Interval of convergence: $(-\infty, \infty)$

Radius of convergence: $R = \infty$.

Problem. For each of the following power series, find the interval and radius of convergence.

a. $\sum_{n=1}^{\infty} \frac{(5x)^n}{n}$

b. $\sum_{n=1}^{\infty} (-1)^n \frac{(x-2)^n}{\sqrt{n}5^n}$

c. $\sum_{n=1}^{\infty} (-1)^n \frac{(x+3)^n}{n^2 2^n}$

d. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

e. $\sum_{n=0}^{\infty} n! x^n$

(a) $$\lim_{n\to\infty} \left| \frac{\frac{(5x)^{n+1}}{n+1}}{\frac{(5x)^n}{n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(5x)^{n+1}}{(5x)^n} \cdot \frac{n}{n+1} \right|$$

$$= |5x|$$

By the ratio test, the series converges when $|5x| < 1$ or $|x| < \frac{1}{5}$ or $-\frac{1}{5} < x < \frac{1}{5}$.

We must test $x = \pm\frac{1}{5}$ separately.

When $x = -\frac{1}{5}$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges

and $x = \frac{1}{5}$, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

So the interval of convergence is

$\left[-\frac{1}{5}, \frac{1}{5}\right)$, $R = \frac{1}{5}$

(b) $$\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{\sqrt{n+1} \cdot 5^{n+1}} \cdot \frac{\sqrt{n} \cdot 5^n}{(x-2)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x-2}{5} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} \right|$$

$$= \left| \frac{x-2}{5} \right|$$

By ratio test, the series converges when

$$\left| \frac{x-2}{5} \right| < 1$$

$$-1 < \frac{x-2}{5} < 1$$

$$-5 < x-2 < 5$$

$$-3 < x < 7$$

We'll test the endpoints separately:

$\underline{x = -3}$ $$\sum_{n=1}^{\infty} (-1)^n \frac{(-5)^n}{\sqrt{n} \cdot 5^n} = \sum_{n=1}^{\infty} (-1)^n \frac{(-1)^n \cdot 5^n}{\sqrt{n} \cdot 5^n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges } (p = 1/2).$$

$\underline{x = 7}$ $$\sum_{n=1}^{\infty} (-1)^n \frac{5^n}{\sqrt{n} \cdot 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}},$$ converges by alt. series test.

Interval: $(-3, 7]$, $R = 5$

ⓒ $\lim_{n\to\infty} \left| \frac{(x+3)^{n+1}}{(n+1)^2 2^{n+1}} \cdot \frac{n^2 \cdot 2^n}{(x+3)^n} \right|$

$= \lim_{n\to\infty} \left| \frac{x+3}{2} \cdot \frac{n^2}{(n+1)^2} \right|$

$= \left| \frac{x+3}{2} \right|$

By ratio test,
the series will converge when

$\left| \frac{x+3}{2} \right| < 1$

$-1 < \frac{x+3}{2} < 1$

$-2 < x+3 < 2$

$-5 < x < -1$

$\underline{x=-5}$

$\sum_{n=1}^{\infty} (-1)^n \frac{(-5+3)^n}{n^2 \cdot 2^n}$

$= \sum_{n=1}^{\infty} (-1)^n \frac{(-1)^n \cdot 2^n}{n^2 \cdot 2^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

converges $p=2>1$.

$\underline{x=-1}$

$\sum_{n=1}^{\infty} (-1)^n \frac{(-1+3)^n}{n^2 \cdot 2^n} = \sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n^2 \cdot 2^n}$

$= \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n^2}$ converges by absolute convergence test.

interval: $[-5,-1]$ $R=2$.

ⓓ $\lim_{n\to\infty} \left| \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right|$

$= \lim_{n\to\infty} \left| \frac{x^2}{(2n+3)(2n+2)} \right|$

$= 0 < 1$

So the series converges for all x

interval: $(-\infty, \infty)$, $R=\infty$.

ⓔ $\lim_{n\to\infty} \left| \frac{(n+1)!\, x^{n+1}}{n!\, x^n} \right| = \lim_{n\to\infty} |x|\, n$

$= \begin{cases} 0 & \text{when } x=0 \\ \infty & \text{when } x\neq 0 \end{cases}$

Series converges when $x=0$ only.

Interval: $\{0\}$,

radius: $R=0$