8.2 Infinite series, continued

Last time We learned

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$
$$= \frac{a(1-r^n)}{1-r} \quad \text{when } r \neq 1$$

Theorem Consider the geometric series $\sum_{k=0}^{\infty} ar^k$.

Then

1. the infinite series converges if and only if $|r| < 1$

2. when $|r|<1$, the infinite series converges to $\frac{a}{1-r}$.

note the starting index is 0, this affects the formula

Proof Notice $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$

$$= a \cdot \lim_{n\to\infty} \frac{1-r^n}{1-r} = \frac{a}{1-r} \quad \text{when } |r|<1$$

Also $\lim_{n\to\infty} S_n$ does not exist when $|r| \geq 1$.

Examples (from Friday's worksheet)

Problem 1

ⓐ $\left(\frac{1}{3}\right)^4 + \left(\frac{1}{3}\right)^5 + \left(\frac{1}{3}\right)^6 + \cdots = \dfrac{(1/3)^4}{1-1/3}$

This is geometric since the ratio of successive terms is constant $r = \frac{1}{3}$

ⓑ $5 - \frac{5}{4} + \frac{5}{4^2} - \frac{5}{4^3} + \frac{5}{4^4} - \cdots = \dfrac{5}{1-\left(-\frac{1}{4}\right)}$

This is geometric since the ratio of successive terms is constant $r = -\frac{1}{4}$

ⓒ $20 + \underbrace{\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \cdots}_{\text{geometric with } a = \frac{1}{3},\ r = \frac{1}{3}} = 20 + \dfrac{1/3}{1-1/3}$

The whole series isn't geometric, but ignoring the first term, it is.

Problem 2

(a) $\lim_{n \to \infty} \left(\frac{5}{6}\right)^n = 0$,

$$\sum_{n=0}^{\infty} \left(\frac{5}{6}\right)^n = \underbrace{1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \cdots}_{\text{geometric with } a=1,\ r=\frac{5}{6}} = \frac{1}{1-5/6} = 6$$

(b) $\lim_{n \to \infty} \left(\frac{-3}{4}\right)^n = 0$

$$\sum_{n=0}^{\infty} \left(\frac{-3}{4}\right)^n = \underbrace{1 - \frac{3}{4} + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^3 + \cdots}_{\text{geometric with } a=1,\ r=\frac{-3}{4}}$$

$$= \frac{1}{1-\left(\frac{-3}{4}\right)} = \frac{4}{7}$$

Problem 3 Does $\sum_{n=1}^{\infty} a_n$ converge?

Hint: Does $\lim_{n\to\infty} s_n = \lim_{n\to\infty} (a_1 + a_2 + a_3 + \dots a_n)$ exist? (Think intuitively)

(a) $a_n = \frac{1}{2}$ $\quad \lim_{n\to\infty} (a_1 + a_2 + \dots + a_n)$

$$= \lim_{n\to\infty} (\underbrace{\tfrac{1}{2} + \tfrac{1}{2} + \dots + \tfrac{1}{2}}_{n \text{ terms}})$$

$$= \lim_{n\to\infty} \tfrac{1}{2} n = +\infty$$

So $\sum_{n=1}^{\infty} a_n$ diverges

(b) $a_n = \dfrac{2n^3 + n + 3}{5n^3 + n^2 + 4n + 6} \approx \dfrac{2}{5}$ when gets large

$$\lim_{n\to\infty} (\underbrace{a_1 + a_2 + \dots + a_n}_{\text{most of these terms are going to be } \approx \frac{2}{5}})$$

$$= \lim_{n\to\infty} \tfrac{2}{5} n = +\infty$$

So $\sum_{n=1}^{\infty} a_n$ diverges

Theorem (n^{th} term test for divergence)

Consider the series $\sum_{n=1}^{\infty} a_n$. If $\lim_{n\to\infty} a_n \neq 0$

then $\sum_{n=1}^{\infty} a_n$ diverges.

Example For each a_n below.

(1) Find $\lim_{n\to\infty} a_n$ and (2) discuss whether

the infinite series $\sum_{n=1}^{\infty} a_n$ converges using nth term test

a) $a_n = 1$ $\lim_{n\to\infty} a_n = 1$, so

series $\sum_{n=1}^{\infty} a_n$ diverges by nth term test

b) $a_n = (-1)^n$ $\lim_{n\to\infty} a_n$ DNE,

so $\sum_{n=1}^{\infty} a_n$ diverges

c) $a_n = 1 - \frac{1}{n}$ $\lim_{n\to\infty} a_n = 1$, so

series $\sum_{n=1}^{\infty} a_n$ diverges

d) $a_n = \frac{1}{n}$ $\lim_{n\to\infty} a_n = 0$, but

we cannot make a conclusion from

nth term test

Question If $\lim_{n\to\infty} a_n = 0$, does that mean that $\sum_{n=1}^{\infty} a_n$ converges? (This is called the converse of the theorem above)

Answer we'll soon see that the answer is not necessarily.

Example Let $a_n = \frac{1}{n}$. What is $\lim_{n\to\infty} a_n$?

Does $\sum_{n=1}^{\infty} a_n$ converge or diverge?